X smooth projective variety over $k = \bar{k}$, $D$ a divisor.

Theorem: There are constants $C_i$ such that for large divisible $m$, $C_1 m^\kappa < h^0(X, mD) < C_2 m^\kappa$.

This $\kappa$ is the Iitaka dimension of $D$.

Key case is Kodaira dimension $D = K_X$. 
Numerical invariance

- This is not a numerical invariant: it can happen that \( D \equiv D' \) but different Iitaka dimension.

- Basic example: torsion and non-torsion classes on an elliptic curve.

- On threefold: two numerically equivalent divisors, one rigid and one which moves in a pencil.

- We want a numerically invariant version \( \nu \)
Numerical dimension: sections

- Fix sufficiently ample $A$.

- Look at growth of $h^0(\lfloor mD \rfloor + A)$, extract a number:

These are numerical invariants. Bonus: makes sense for $R$-divisors. Roundup vs down doesn't matter.
Fix sufficiently ample $A$.

Look at growth of $h^0(\lfloor mD \rfloor + A)$, extract a number:

$$\kappa^+_\sigma(D) = \min \left\{ k : \limsup_{m \to \infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} < \infty \right\}$$

$$\kappa^-\sigma(D) = \max \left\{ k : \limsup_{m \to \infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} > 0 \right\}$$

$$\kappa^-\sigma(D) = \max \left\{ k : \liminf_{m \to \infty} \frac{h^0(\lfloor mD \rfloor + A)}{m^k} > 0 \right\}$$

These are numerical invariants.

Bonus: makes sense for $\mathbb{R}$-divisors. Roundup vs down doesn’t matter.
Volume of $D$ is $\lim_{m \to \infty} \frac{h^0(mD)}{m^d/d!}$.

Asymptotic growth rate of number of sections.

Extends to a continuous function on $\overline{\text{Eff}}(X)$. 
An example

- $X$ the blow-up of $\mathbb{P}^2$ at 9 very general points.

- Then $h^0(-mK_X) = 1$, but $h^0(-mK_X + A) \sim Cm$.

- So $\kappa(-K_X) = 0$ but $\kappa_\sigma(-K_X) = 1$.

- Abundance: $\kappa(K_X) = \kappa_\sigma(K_X)$ for any klt $X$. 
Proposition 3.3.2. Let $X$ be a smooth projective variety and let $D$ be a pseudo-effective $\mathbb{R}$-divisor. Let $B$ be any big $\mathbb{R}$-divisor.

If $D$ is not numerically equivalent to $N_\sigma(D)$, then there is a positive integer $k$ and a positive rational number $\beta$ such that

$$h^0(X, \mathcal{O}_X(\lceil mD \rceil + \lceil kB \rceil)) > \beta m, \quad \text{for all} \quad m \gg 0.$$ 

Proof. Let $A$ be any integral divisor. Then we may find a positive integer $k$ such that

$$h^0(X, \mathcal{O}_X(\lceil kB \rceil - A)) \geq 0.$$ 

Thus it suffices to exhibit an ample divisor $A$ and a positive rational number $\beta$ such that

$$h^0(X, \mathcal{O}_X(\lceil mD \rceil + A)) > \beta m \quad \text{for all} \quad m \gg 0.$$ 

Replacing $D$ by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply (V.1.12) of [28]. \qed

27
Conjecture (Nakayama, 2002)

Theorem (2012-2016, 2016-2019)

Suppose that $D$ is a pseudoeffective divisor and that $A$ is ample. Then there exist constants $C_1, C_2$ and a positive integer $\nu(D)$ so that:

$$C_1 m^{\nu(D)} \leq h^0(\lfloor mD \rfloor + A) \leq C_2 m^{\nu(D)}$$

- We are going to see that this is false!

- As far as I know, all uses in literature have been corrected by Fujino ’20.
Some definitions

- $N^1(X) = N^1(X) \otimes \mathbb{R}$ is divisors on $X$ modulo numerical equivalence.

- $\text{Eff}(X)$ is the cone in $N^1(X)$ spanned by classes of effective divisors.

- $\overline{\text{Eff}}(X)$ is the closure of this cone.
General remarks

- Suppose $X$ is a variety and $\phi : X \to X$ is a pseudoautomorphism (i.e. isomorphism in codimension 1).

- $\phi^*$ preserves $h^0(L)$, hence volume, as well as various positive cones.

- Look at $\phi^* : N^1(X) \to N^1(X)$, assume $\lambda_1 > 1$.

- Perron–Frobenius: the eigenvector $D_+$ lies on the boundary of $\overline{\text{Eff}}(X)$ (and $\overline{\text{Mov}}(X)$).
First example

- Let $X$ be $(1, 1), (1, 1), (2, 2)$ complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$.

- This is a smooth CY3, Picard rank 2.

- Studied by Oguiso in connection with Kawamata–Morrison conj.
The example

- It has some birational automorphisms coming from covering involutions.

- Action on $N^1(X)$ given by

  $$\tau_1^* = \begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}, \quad \tau_2^* = \begin{pmatrix} -1 & 0 \\ 6 & 1 \end{pmatrix}, \quad \phi^* = \begin{pmatrix} 35 & 6 \\ -6 & -1 \end{pmatrix}$$

- Composition has infinite order: $\lambda = 17 + 12\sqrt{2}$.

- Nef cone bounded by $H_1, H_2$.

- Psef cone bounded by $(1 \pm \sqrt{2})H_1 + (1 \mp \sqrt{2})H_2$.

- Let $D_1 = c_1 H_1 + c_2 H_2$ be divisor in this class.
Here’s the good news. For any line bundle whatsoever on $X$, you can compute $h^0(D)$.

Pull it back some number of times, it’s ample, and then compute $h^0$ for ample using Riemann-Roch+Kodaira vanishing!

HRR on CY3:

$$\chi(D) = \frac{D^3}{6} + \frac{D \cdot c_2(X)}{12}.$$
Let’s do it!

- Suppose our ample is $A = M_1 D_1 + M_2 D_2$.

- We need to compute $h^0([mD] + A)$.

- How many times to pull back? Looks like a mess, but there’s an invariant quadratic form: the product of the coefficients when you work in the eigenbasis.

$$\phi^* = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$
So $mD_1 + A$ is $(m + M_1, M_2)$.

The pullback that’s ample has the two coefficients roughly equal:

$$\left( \sqrt{(m + M_1)M_2}, \sqrt{(m + M_1)M_2} \right)$$

Then $h^0(\lfloor mD_1 \rfloor + A) \approx Cm^{3/2}$. 
Let $X$ be a hypersurface in $(\mathbb{P}^1)^4$ of bidegree $(2, 2, 2, 2)$. This is a Calabi–Yau threefold, satisfying the Kawamata–Morrison conjecture [Cantat–Oguiso].

$\text{Bir}(X) \cong (\mathbb{Z}/2\mathbb{Z})^4$

What do the positive cones of divisors look like?
Two kinds of divisors on the psef boundary

- There are some distinguished semiample classes:
  \[ \pi_i^* (\mathcal{O}_{\mathbb{P}^1}(1)) + \pi_j^* (\mathcal{O}_{\mathbb{P}^1}(1)) \].

- Similarly, orbit of these classes under Bir(\(X\)). We’ll call these “semiample type”. (NB: they aren’t semiample.)

- All these have \( \text{vol}(D + tA) \sim Ct \).

- There are also many eigenvector classes:

- All these have \( \text{vol}(D + tA) \sim Ct^{3/2} \).
First picture: the eigenvectors
Plotting those divisors

Blue are eigenvectors, red are semiample type.
Plotting those divisors

Blue are eigenvectors, red are semiample type.
Volume near the boundary

- There are some “circles” on the boundary of \( \overline{\text{Eff}}(X) \) on which both eigenvectors and semiample type are dense.
- The former have \( \text{vol}(D + tA) \sim Ct^{3/2} \), the latter have \( \text{vol}(D + tA) \sim Ct \).
- How is this possible?
- Because the volume function is so easy to compute numerically, we can plot it!
Volume near the boundary
Cross-section: $\epsilon = 0.002$

- **y-axis** is $\frac{\log h^0(\text{vol}(D_t + \epsilon A))}{\log \epsilon}$, the estimated exponent of the volume (not all the way down to 1 because of constants)
Cross-section: $\epsilon = 0.0005$
Cross-section: $\epsilon = 5 \cdot 10^{-5}$
Cross-section: $\epsilon = 5 \cdot 10^{-6}$
Theorem (probably)

There exist classes $D$ on $\overline{\text{Eff}}(X)$ for which $\kappa_{\sigma}^-(D) = \frac{3}{2}$ and $\kappa_{\sigma}^+(D) = 2$.

This means roughly that $h^0(\lfloor mD \rfloor + A)$ oscillates between $C_1 m^{3/2}$ and $C_2 m^2$ as $m$ increases.
Strategy

- If $D$ is “close” to one of our semiample type classes, then $\text{vol}(D + \epsilon A)$ will be larger than expected as long as $\epsilon$ is not too small.

- But as $\epsilon$ shrinks, $D$ isn’t close enough any more.

- Goal: find divisors which are unusually close to semiample type at many different scales

- Analogous to Liouville type numbers $\sum_{k=1}^{\infty} \frac{1}{10^k!}$ which are close to rationals at many scales.

- Behavior of $h^0(\lfloor mD \rfloor + A)$ depends in a subtle way on Diophantine properties of $D$. 
Location of the “hills”

(This is roughly a plot of all classes whose estimated exponent is < 1.4)
A divisor with some oscillation
Choose a pseudoautomorphism $\phi : X \to X$ with $\lambda = 1$ Jordan block whose leading eigenvector is a semiample with exponent 1.

Choose a pseudoautomorphism $\psi : X \to X$ with $\lambda > 1$, so leading eigenvector $D_1$ has $3/2$ exponent.

Choose a rapidly increasing sequence $n_i = n!$ (or maybe even more).

Fix an ample $A$ and consider the sequence of divisors:
Consider the following sequence of divisors, normalized to have length 1:

- $(\phi^*)^{n_1} A$
- $(\phi^*)^{n_1} (\psi^*)^{n_2} A$
- $(\phi^*)^{n_1} (\psi^*)^{n_2} (\phi^*)^{n_3} A$
- ... 

After rescaling, these converge to a limit $D_+$. 
Why it works

- \((\phi^*)^n A\) will be within \(\epsilon\) of a semiample type if \(n_3 \gg 0\).

- \((\phi^*)^{n_1}(\psi^*)^{n_2}(\phi^*)^{n_3} A\) is still within \(\epsilon\) of a semiample if \(n_3 \gg n_2\).

- That class is also very close to the limit \(D_+\).

- So \(D_+\) is close enough to a divisor of semiample type to get an increase in the volume at a suitable \(\epsilon\) scale.

- Similarly, \(D_+\) is close enough to eigenvectors at suitable scales to get exponent near \(3/2\).