SINGULARITIES IN PRIME CHARACTERISTIC

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1. Introduction

Reference:

(1) *Frobenius splitting in commutative algebra*, Karen Smith and Wenliang Zhang.
(2) *A survey of test ideals*, Karl Schwede and Kevin Tucker.
(3) *Globally F-regular and log Fano varities*, Karl Schwede and Karen Smith.
(4) *Characterizations of regular local rings of characteristic p*, Ernst Kunz.
(5) *On Noetherian rings of characteristic p*, Ernst Kunz.
(6) *F-purity and rational singularity*, Richard Fedder.
(7) *'Varieties'* and *'Algebraic curves'*, Stacks Project.

Things we won't talk about much, or at all:

- Reduction to char. p.
- Tight closure.
- Application to invariant theory.
- Hilbert-Kunz multiplicity.
- F-signature.

Warning: There are going to be many results stated without proofs. This does not mean the proofs are difficult. In fact, many proofs, such as that of Theorem 2.2.2, are quite simple.

These notes have not been proofread. So please use them with that knowledge. If anything is confusing, please email me.

2. Lecture 1- Basic Setup

For the rest of the course, we will fix a ring R of prime char. $p > 0$. We will also assume that R is non-trivial. Note in this case R contains the field $\mathbb{F}_p$. Thus, R is equicharacteristic.

A local ring for us is not necessarily Noetherian. We won't just stick to a Noetherian setting. I like valuation rings, so we will use these as examples. The material of this mini-course is usually developed with Noetherian hypotheses.

Char. p is nice because:

- commutative algebra is nice when rings contain a field. In fact, many important conjectures in commutative algebra have been worked out in the equicharacteristic case, but are remain open in mixed characteristic.
- Although we lose geometric results like resolution of singularities and Kodaira vanishing, we gain the Frobenius endomorphism. Surprisingly, the Frobenius map detects singularities!

2.1. Enter Frobenius.

Freshman's dream gives us the ring homomorphism

$$F : R \to R$$
$$r \mapsto r^p$$

called the **Frobenius map**. We can also iterate the Frobenius map getting

$$F^e : R \to R.$$
$$r \mapsto r^{p^e}$$

Exercise 2.1.1. Let $F : R \to R$ be the Frobenius map. Show that the induced map on Spec is the identity map on the underlying topological spaces.

Using the previous exercise, we can define

Definition 2.1.2. Let X be a scheme over $\mathbb{F}_p$. The **(global) Frobenius morphism**

$$F : X \to X$$

is the morphism that is the identity on the underlying topological spaces, and such that the induced map of sheaves $\mathcal{O}_X \to F_*\mathcal{O}_X$ raises sections to the p^{th} power. In particular, F^e is an affine morphism.

Notation 2.1.3. Following sheaf notation, we will denote the target copy of R in $F : R \to R$ with module structure induced via restriction of scalars as F_*R. For the e^{th}-iterate of the Frobenius, this becomes F^e_*R. To spell it all out, this means that as a ring, F^e_*R is just R. However, as an R module, for $r \in R$ and $x \in F^e_*R$, $r \cdot x = r^{p^e}x$.

Exercise 2.1.4. Show the following:

1. F^e_* is an exact functor $Mod_R \to Mod_R$. Note F^e_* is a restriction of scalars functor. Do not confuse it with the base change functor $F^e_*R \otimes_R \_$.
2. For an R-module M, $F^f_*(F^e_*M) = F^{e+f}_*M$.

(3) If S is a multiplicative subset of R, then show that $S^{-1}F^e_*R \cong F^e_*(S^{-1}R)$ as $S^{-1}R$-modules.

2.2. **Kunz's theorem and F-finiteness.** We want to define and study various notions using iterates of Frobenius that test how far a ring is from being regular. Globally, we want to define notions to detect how far a locally Noetherian scheme over $\mathbb{F}_p$ is from being regular/non-singular [1]. I will assume you are familiar with regular local rings. However, recall that a regular local ring is

- a domain.
- a UFD.
- Gorenstein.
- Cohen Macaulay
- completion of a regular local ring containing a field is $\cong k[[x_1, \ldots, x_n]]$ for a field k (in fact, k is isomorphic to the residue field of the local ring).

Frobenius detects one of the most basic singularities already:

Exercise 2.2.1. (**Frobenius detects reducedness**) Show that a scheme X over $\mathbb{F}_p$ is reduced if and only if the induced map on sheaves $F^\sharp : \mathcal{O}_X \to F_*\mathcal{O}_X$ is injective.

The starting point of the use of Frobenius to detect singularities is the following result due to Kunz:

Theorem 2.2.2 (Kunz '69)**.** *Let R be a Noetherian ring of characteristic $p > 0$. Then R is regular if and only if the Frobenius map $F : R \to R$ is flat.*

We will not prove this theorem. Recall that a Noetherian (not necessarily local) ring is regular if the localization of the ring at all primes is a regular local ring in the usual sense. The global translation of Kunz's result is that a locally Noetherian scheme X over $\mathbb{F}_p$ is a regular/non-singular if the Frobenius morphism $F : X \to X$ is flat. Thus, regularity of a locally Noetherian scheme implies that $F_*\mathcal{O}_X$ is an $\mathcal{O}_X$-vector bundle if in addition the Frobenius morphism F is a finite morphism, as can be seen with the help of the following exercise:

Exercise 2.2.3. Show the following:

(1) For a finitely generated module M over a Noetherian local ring R, M is flat if and only if M is free over R.
(2) Let X be a locally Noetherian scheme over $\mathbb{F}_p$ such that the Frobenius morphism $F : X \to X$ is finite. Then X is regular at a point $x \in X$ if and only if $F_*\mathcal{O}_X$ is a free $\mathcal{O}_X$-module in neighborhood of x. Thus, X is regular if and only if $F_*\mathcal{O}_X$ is a locally free $\mathcal{O}_X$-module.
(3) If X is an in (2), then the regular locus of X is open.

Exercise 2.2.4. (Fun easy exercise) Use Kunz's theorem to give a non homological proof of the fact that regularity of a Noetherian ring of prime characteristic is preserved under localization!

[1]A locally Noetherian scheme is **regular/non-singular** if all its stalks are regular local rings.

Exercise 2.2.5. Show that if R is a regular ring of prime characteristic, then the polynomial ring $R[X_1, \ldots, X_n]$ is also regular. *Hint: Use Kunz's theorem.*

Finiteness of the Frobenius morphism is not a very strong restriction, and in most cases in geometry the Frobenius morphism will be finite. So let us introduce the following terminology:

Definition 2.2.6. Let X be a scheme over $\mathbb{F}_p$. We say that X is **F-finite** if the Frobenius morphism $F : X \to X$ is a finite morphism. We say a ring R of char. p is **F-finite** if $\operatorname{Spec}(R)$ is F-finite.

Exercise 2.2.7. Show the following rings are F-finite:

(1) A perfect field.
(2) Homomorphic image of an F-finite ring.
(3) Finitely generated algebras over F-finite rings.
(4) Localization of an F-finite ring at a multiplicative set.
(5) If the residue field of a complete local ring is F-finite, then the ring is F-finite (Hint: Cohen's structure theorem for equicharacteristic complete local rings).
(6) Completion of an F-finite, Noetherian, local ring at its maximal ideal is F-finite.

The previous exercise shows that any scheme locally of finite type over a perfect field of prime characteristic will be F-finite. Hence, F-finiteness is not a very strong restriction.

For an F-finite Noetherian local ring R, Kunz's theorem and Exercise 2.2.3 show that R is regular if and only if F_*R is a free R-module (necessarily of finite rank). One also shows in Exercise 2.2.3 is that the regular locus of an F-finite, locally Noetherian scheme is open. The notions of singularity we will introduce in this course all measure how far F_*R is from being a free R-module.

Another nice consequence of F-finiteness is the following result, also due to Kunz:

Theorem 2.2.8 (Kunz '76)**.** *Let R be a Noetherian F-finite ring. Then R has finite Krull dimension.*

2.3. F-finiteness in valuation rings.

F-finiteness in valuation rings. Although F-finiteness is most useful with some Noetherian hypotheses, the story is surprisingly pretty when we ask which valuation rings of characteristic p are F-finite. Recall,

Definition 2.3.1. If K is a field, a subring V of K is a **valuation ring** of K if for all $x \in K$, $x \in K$ or $x^{-1} \in K$.

Can you use the above definition and some elementary algebra to show the following:

Exercise 2.3.2. Let V be a valuation ring of a field K. If I, J are ideals of V, then $I \subseteq J$ or $J \subset I$. In particular, for any $a, b \in V$, $a|b$ or $b|a$. [2]

[2]In fact, if the ideals of a domain are linearly ordered by inclusion, then the domain must be a valuation ring of its fraction field. But this takes a little work.

The previous exercise shows that the spectrum of a valuation ring consists of a single chain of prime ideals. The union of this chain of prime ideals is the unique maximal ideal of the valuation ring, i.e., valuation rings are local. When a valuation ring V has finite Krull dimension, one can think of morphisms $Spec(V) \to X$ (for a scheme X) to be like the analogue of a path on X.

Exercise 2.3.3. (Noetherian valuation rings) In this exercise we will figure out which valuation rings are Noetherian.

(1) Show that any finitely generated ideal of a valuation ring must be principal.
(2) Show that a Noetherian valuation ring must be a DVR (discrete valuation ring), i.e., a regular local ring of dimension 1.

The total ordering of ideals of a valuation ring has strong implications for algebraic properties of the ring. Here is one surprisingly property that will be very important in this subsection:

Exercise 2.3.4. Let V be a valuation ring and M a V-module.

(1) If M is finitely generated and torsion free, show that M is free *Hint: take a minimal set of generators of M, assume they have a non-trivial relation and then use Exercise 2.3.2 to get a contradiction.*
(2) Show that if M is torsion free (but not necessarily finitely generated), then M is flat *Hint: Use the fact that a filtered direct limit of flat modules is flat.*

Use the previous exercise to now prove the following:

Theorem 2.3.5 (Datta-Smith '15). *Let V be a valuation ring of prime characteristic. Show that the Frobenius map $F : V \to V$ is flat. In particular, if V is F-finite, then F_*V is a free V-module.*

Remark 2.3.6. Recall that flatness of Frobenius characterizes regular local rings among Noetherian local rings of prime characteristic. Since Frobenius is always flat for valuation rings, in some sense valuation rings in prime characteristic can be thought of as non-Noetherian analogues of regular local rings.

2.4. **F-finiteness and excellence.** There is a very close relationship between F-finite Noetherian rings and excellence, further providing evidence that F-finiteness is a very geometric notion. It is not important for us to know the precise definition of an excellent ring (it is a bunch of technical, but easy to understand conditions).

Excellent rings were introduced by Grothendieck because arbitrary Noetherian rings can be quite pathological. For example, the integral closure of a Noetherian domain in a finite field extension of its fraction field may fail to be module finite over the original domain. Similarly, given two prime ideals $p \subsetneq q$ of a Noetherian ring R, there can be saturated chains of prime ideals from p to q of different lengths (i.e. an arbitrary Noetherian ring can fail to be *catenary*). However, the class of excellent rings don't have such pathological properties.

Remark 2.4.1. The class of excellent rings is closed under taking finite type algebras and localization. Moreover, all complete local rings are excellent (any characteristic). Furthermore, Dedekind domains of characteristic zero are excellent. In particular, any localization of a finite type algebra over a field is excellent.

Kunz first established the important connection between F-finiteness and excellence:

Theorem 2.4.2. *Let R be a F-finite Noetherian ring of prime characteristic. Then R is excellent.*

The converse of the above theorem is true under mild additional hypotheses:

Exercise 2.4.3. Suppose R is a Noetherian, excellent domain of prime characteristic whose fraction field is F-finite. Then show that R is F-finite. *Hint: Use the fact that for an excellent domain R, the integral closure of R in a finite extension of Frac(R) is module finite over R.*

Exercise 2.4.4. Can you give an example of an excellent ring of prime characteristic that is not F-finite? *Hint: The coordinate ring of a suitable affine line over a field will do.*

The previous exercise, combined with Theorem 2.4.2 shows that in a function field over an F-finite ground field (like a perfect ground field), an excellent Noetherian subring is the same as an F-finite, Noetherian subring.

We now give an example of a regular local subring V of $\mathbb{F}_p(x,y)$ (with fraction field $=$ $\mathbb{F}_p(x,y)$) that is not excellent, hence by the above exercise also not F-finite. In particular, this gives an example of a regular local ring for which Frobenius is flat, but for which F_*V is not a free V-module.

Example 2.4.5. (A DVR in a function field that is not excellent.) Consider the Laurent series field $\mathbb{F}_p((t))$ with the usual discrete valuation

$$v : \mathbb{F}_p((t))^\times \to \mathbb{Z}$$

whose corresponding valuation ring is the power series DVR $\mathbb{F}_p[[t]]$. Clearly, $\mathbb{F}_p((t))$ is uncountable, whereas $\mathbb{F}_p(t)$ is countable. So, there exists a power series $p(t) \in \mathbb{F}_p((t))$ such that $t, p(t)$ are algebraically independent over $\mathbb{F}_p$ [3]. This gives us a homomorphism of fields

$$\mathbb{F}_p(x,y) \hookrightarrow \mathbb{F}_p((t)).$$
$$x \mapsto t, y \mapsto p(t)$$

Restricting the valuation v to $\mathbb{F}_p(x,y)$ via the above inclusion of fields produces a valuation on $\mathbb{F}_p(x,y)$ whose corresponding valuation ring V is the desired non-excellent, non F-finite DVR. Now you can either directly show that V is a non-excellent DVR if you know the definition of excellent rings, or you can use valuation theoretic techniques developed in [Datta-Smith] to show that V is not F-finite, hence also not excellent (use Exercise 2.4.3 for the latter approach).

[3] If L is algebraic over a field K, then L and K have the same cardinality.

3. Lecture 2- Frobenius splitting and purity

In the previous lecture, we stated that if R is a Noetherian local and F-finite ring, then R is regular if and only if F_*R is a free R-module. This means regularity of a Noetherian local ring R is characterized by a decomposition of F_*R as a direct sum of copies of R. What happens when we relax this condition? That is, what if we require F_*R to have at least one free R summand? This leads to the notion of Frobenius splitting, and the related weaker notion of F-purity.

3.1. Local Frobenius splitting and F-purity.

Definition 3.1.1. Let R be a ring of prime characteristic. Then R is **Frobenius split** if the Frobenius map $F : R \to F_*R$ has a left inverse in Mod_R, i.e., if there exists an R-linear map $\varphi : F_*R \to R$ such that the composition

$$R \xrightarrow{F} F_*R \xrightarrow{\varphi} R$$

is the identity.

Exercise 3.1.2. (Easy exercise) Show that the existence of a left inverse of F is equivalent to the existence of an R-linear map $F_*R \to R$ that maps $1 \mapsto 1$.

Exercise 3.1.3. (Reality check) If R is Frobenius split, then R is reduced. Moreover, R is Frobenius split if and only if the induced map $\mathrm{Hom}_R(F_*R, R) \xrightarrow{-\circ F} \mathrm{Hom}_R(R, R)$ is surjective.

Exercise 3.1.4. (Frobenius splitting is a local property for Noetherian F-finite rings) Let R be a Noetherian, F-finite ring.

(1) If R is F-split, then for any multiplicative subset S, $S^{-1}R$ is F-split.
(2) If for some $q \in \mathrm{Spec}(R)$, R_q is F-split, then there exists an open neighborhood $D(f)$ of q such that R_f is F-split. Hence show that the F-split locus of R in $\mathrm{Spec}(R)$ is open.
(3) Show that R is Frobenius split if and only if for all prime ideals $p \subset R$, R_p is Froebnius split.
(4) Show that an F-finite regular ring is Frobenius split.

Hint: Use the interpretation of splitting as surjectivity of suitable Hom modules as in Exercise 3.1.3. Why do we need F-finiteness?

Let S be a finitely generated $\mathbb{N}$-graded ring over a field with irrelevant ideal m. Often S satisfies a property P if and only if S_m satisfies P. For instance, P could be the property of S being normal (when S is a domain) or Cohen-Macaulay. The same holds for Frobenius splitting for F-finite graded rings.

Proposition 3.1.5. *Let S be a $\mathbb{N}$-graded ring, finitely generated over an F-finite field k (this ensures S is F-finite). Let m be the irrelevant ideal of S. Then*

(1) *The non Frobenius split closed locus of Spec(S) is defined by a homogenous ideal.*
(2) *S is Frobenius split if and only if S_m is F-split.*

Proof. Note (2) follows from (1). For if S_m is F-split, then the homogenous ideal I defining the non F-split closed locus of $\mathrm{Spec}(S)$ has to be the whole ring S as otherwise it must be contained in m.

We omit the proof of (1). However, note that when k is infinite, we can obtain a particularly simple proof of (1) using the $k^{\times}$ action on S (what is this action?). The point is we want to show that the unique radical ideal $\mathcal{I}$ of S defining the non Frobenius split closed locus of S is homogenous– each element of $k^{\times}$ determines a ring automorphism of S that clearly maps $\mathcal{I}$ to itself. When the field k is infinite, this proves I is homogenous (think about this)! □

3.2. **Pure maps.** There is in fact a notion weaker than Frobenius splitting, called F-purity. In a wide range of cases, Frobenius splitting and F-purity coincide which is why many algebraists/geometers use the term F-purity and Frobenius splitting interchangeably. But I think this is bad practice, so we will always differentiate between purity and splitting in this lecture series.

To define F-purity, we first need to introduce the notion of a pure map of modules.

Definition 3.2.1. Let $\varphi : M \to N$ be a map of R-modules. Then φ is a **pure** map if for all R-modules P, the induced map

$$\varphi \otimes id : M \otimes_R P \to M \otimes_R P$$

is injective.

Here is an easy exercise:

Exercise 3.2.2. (Reality check) Let $M \to N$ be a map of R-modules that has a left inverse in Mod_R. Then show that $M \to N$ is pure. Moreover, show that a pure map of modules is injective.

Lemma 3.2.3. *Let A be an arbitrary commutative ring, not necessarily Noetherian nor of characteristic p.*

(a) *If $M \to N$ and $N \to Q$ are pure maps of A-modules, then the composition $M \to N \to Q$ is also pure.*
(b) *If a composition $M \to N \to Q$ of A-modules is pure, then $M \to N$ is pure.*
(c) *If B is an A-algebra and $M \to N$ is pure map of A-modules, then $B \otimes_A M \to B \otimes_A N$ is a pure map of B-modules.*
(d) *Let B be an A-algebra. If $M \to N$ is a pure map of B-modules, then it is also pure as a map of A-modules.*
(e) *An A-module map $M \to N$ is pure if and only if for all prime ideals $\mathcal{P} \subset A$, $M_{\mathcal{P}} \to N_{\mathcal{P}}$ is pure.*
(f) *A faithfully flat map of rings is pure.*
(g) *If $(\Lambda, \leq)$ is a directed set with a least element λ_0, and $\{N_\lambda\}_{\lambda \in \Lambda}$ is a direct limit system of A-modules indexed by Λ and $M \to N_{\lambda_0}$ is an A-linear map, then $M \to \varinjlim_\lambda N_\lambda$ is pure if and only if $M \to N_\lambda$ is pure for all λ.*
(h) *A map of modules $A \to N$ over a Noetherian local ring (A, m) is pure if and only if $E \otimes_A A \to E \otimes_A N$ is injective where E is the injective hull of the residue field of R.*

Remark 3.2.4. We will not prove this Lemma. Most of the assertions are easy to prove using the definition of purity, and elementary properties of tensor product. One key property, listed in the above lemma, is that a faithfully flat ring map is pure [4]. Thus, if $(A, \mathfrak{m})$ is a Noetherian local ring and, $\widehat{A}$ denotes its $\mathfrak{m}$-adic completion, then $A \to \widehat{A}$ is pure.

3.3. **F-purity and its relationship with F-splitting.** We next describe the close relationship between splitting of a map of modules and purity. It turns out that in many scenarios splitting and purity coincide, especially with rings and modules arising in geometry,

Proposition 3.3.1 (Hochster-Roberts)**.** *Let A be an arbitrary ring, and $\varphi : M \to N$ a map of R-modules such that* $\operatorname{coker}(\varphi)$ *is finitely presented. Then φ has a left inverse in* Mod_R *if and only if φ is pure.*

As a simple application of the Hochster-Roberts criterion, show that

Exercise 3.3.2. Let R be a Noetherian ring and $\varphi : M \to N$ be a map of R-modules such that N is finitely generated. Then φ splits if and only if φ is pure.

We now come to the next notion of F-singularity which is technically weaker than Frobenius splitting, but coincides with the latter in Noetherian F-finite rings.

Definition 3.3.3. A ring R of prime characteristic is **F-pure** if the Frobenius map $F : R \to F_*R$ is a pure map of R-modules.

Note that since pure maps are injective, R F-pure $\Rightarrow$ R is reduced.

Exercise 3.3.4. Let R be a reduced, F-finite ring (not necessarily Noetherian). Show that R is Frobenius split if and only if R is F-pure. *Hint: Apply Proposition 3.3.1.*

Example 3.3.5. Since Frobenius is flat, hence faithfully flat for valuation rings and regular rings of prime characteristic, these rings are F-pure. We will see later that the non-excellent, non F-finite DVR constructed in Example 2.4.5 is F-pure, but not F-split.

Remark 3.3.6. Many good properties possessed by a Frobenius split ring are already possessed by an F-pure ring. In fact, F-purity seems to be a better notion of singularity than Frobenius splitting in the non-Noetherian, or even Noetherian but non F-finite setting. This is because even regular rings can fail to be Frobenius split when they are not F-finite, but they are always F-pure, and one would certainly want a notion of F-singularity to at least encompass the class of regular rings.

Another reason why F-purity is a better notion without finiteness hypotheses is that F-purity is always a local property, whereas Frobenius splitting may not be a local property without the ring being F-finiteness (please see Exercise 3.1.4).

Exercise 3.3.7. Let R be any ring of prime characteristic. Then R is F-pure if and only if for all prime ideals q, R_q is F-pure. *Hint: Use Lemma 3.2.3(e).*

[4] A ring map $A \to B$ is **faithfully flat** if B is a flat A-module and the induced map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective.

3.4. **Frobenius splitting and F-finiteness.** As discussed in the previous subsection, Frobenius splitting seems to be best behaved under finiteness hypotheses. In this section explore just how intimate a relation it has with F-finiteness.

Theorem 3.4.1 (Datta-Smith'15). *Let R be a Noetherian domain of prime characteristic whose fraction field K is F-finite (such as the function field of a variety over a perfect field). Then the following are equivalent:*

1. *R is F-finite.*
2. *R is excellent.*
3. *There exists $e > 0$, such that* $\mathrm{Hom}_R(F^e_* R, R) \neq 0$.

The proof of the theorem basically uses linear algebra (we will omit it). Since the existence of a splitting of the Frobenius map implies that $\mathrm{Hom}_R(F_* R, R)$ is non-trivial, we immediately get the following result as a consequence:

Theorem 3.4.2 (**Splitting implies F-finiteness**). *Let R be a Noetherian domain with F-finite fraction field. If R is Frobenius split, then it is F-finite.*

Since we mostly work with fields that are F-finite in geometry (the function field over an F-finite field is F-finite), for most Noetherian rings arising in geometry, F-finiteness is a consequence of Frobenius splitting.

Example 3.4.3. (Example of a regular local ring which is F-pure but not F-split) In Example 2.4.5 we constructed a DVR of $\mathbb{F}_p(x, y)$ which is not excellent, hence not F-finite. Thus, this DVR is also not Frobenius split by the above results. However, it is F-pure since Frobenius is faithfully flat for regular rings.

Open Question: As far as I know, no one knows an example of an excellent domain which is F-pure but not F-split.

Remark 3.4.4. Note that the DVR constructed in Example 3.4.3 is not excellent. Also, Theorem 3.4.1 shows that an example cannot be given whose fraction field if F-finite. This is because if we have a Noetherian excellent domain with F-finite fraction field, then by the aforementioned result, the domain is F-finite. But then by the Hochster-Roberts criterion 3.3.1 such a domain will be F-pure if and only if it is F-split.

3.5. **Direct summands of Frobenius split rings.** The next result is quite easy to prove, but is surprisingly useful in establishing that many rings are Frobenius split:

Theorem 3.5.1. *Let $R \hookrightarrow S$ be an inclusion of rings such that R is a direct summand of S, i.e., the inclusion of rings has a left inverse in* Mod_R. *If S is Frobenius split, then so is R.*

Proof. Exercise. □

As an immediate consequence, we obtain:

Corollary 3.5.2. *Let S be an F-finite regular ring (such as a polynomial ring over a perfect field). Then a direct summand of S is Frobenius split.*

Here are some fun applications:

Exercise 3.5.3. (**Veronese subrings**) Let S be an $\mathbb{N}$-graded ring which is Frobenius split (such as a polynomial ring over a field). Show that the Veronese subrings of S are Frobenius split.

Proposition 3.5.4 (**Rings of invariants**)**.** *Let G be a finite group acting on a ring S of prime characteristic p. Assume that p does not divide $|G|$. If S is Frobenius split, then the ring of invariants S^G is also Frobenius split.*

Proof. Exercise: Create a splitting of $S^G \hookrightarrow S$ by 'averaging' the group action. $\square$

4. Lecture 3- Global Frobenius splitting and consequences

4.1. **Introduction.** For a scheme X over $\mathbb{F}_p$, we can require the morphism of sheaves $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ to split locally (i.e. at all stalks), or we can require the morphism to split globally. We will see that global splitting of Frobenius morphism has strong consequences for the geometry of X. But first, let us formally define global splitting:

Definition 4.1.1. A scheme X over $\mathbb{F}_p$ is **(globally) Frobenius split** or **(globally) F-split** if the morphism $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ has a left inverse in the category of $\mathcal{O}_X$-modules.

I am abusing notation a little bit, and denoting the morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ also by F, when I should be using something like $F^\sharp$.

Remark 4.1.2. We will often drop the adjective 'global' from global Frobenius splitting and just call it Frobenius splitting. If we want to only consider local Frobenius splitting, we will explicitly say so.

Exercise 4.1.3. If X is affine with coordinate ring R, show that X is Frobenius split if and only if R is Frobenius split in the sense defined before.

Exercise 4.1.4. Suppose there exists $e > 0$ such that $F^e : \mathcal{O}_X \to F^e_*\mathcal{O}_X$ has a left inverse in the category of $\mathcal{O}_X$-modules. Then show that X is Frobenius split. *Hint: Factor $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ through $F_*\mathcal{O}_X$.*

Exercise 4.1.5. Show that a Frobenius split scheme must be reduced.

Lemma 4.1.6. *Let X be a Frobenius split scheme. Then for all $e > 0$, the morphism $F^e : \mathcal{O}_X \to F^e_*\mathcal{O}_X$ has a left inverse in the category of $\mathcal{O}_X$-modules.*

Proof. We prove this by induction on e. The case $e = 1$ follows because X is Frobenius split. Suppose that the statement holds for e. So $F^e : \mathcal{O}_X \to F^e_*\mathcal{O}_X$ has a left inverse

$$\varphi : F^e_*\mathcal{O}_X \to \mathcal{O}_X.$$

Applying the functor F_*, we see that $F_*\mathcal{O}_X \to F^{e+1}_*\mathcal{O}_X$ then has a left inverse $F_*\varphi$. Now let

$$\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$$

be a left inverse of $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ (such a left inverse exists because X is Frobenius split). Then

$$\phi \circ F_*\varphi$$

is a left inverse of $F^{e+1} : \mathcal{O}_X \to F^{e+1}_*\mathcal{O}_X$. □

4.2. **Local versus global Frobenius splitting for projective schemes.** An affine scheme $\mathrm{Spec}(R)$ is Frobenius split if and only if the ring R is Frobenius split. Let I be a homogeneous ideal of $k[X_0, \ldots, X_n]$, and $X = \mathrm{Proj}(k[X_0, \ldots, X_n]/I)$. What is the relationship between Frobenius splitting of the scheme X and Frobenius splitting of the graded ring $k[X_0, \ldots, X_n]/I$?

Exercise 4.2.1. Let $X = \mathrm{Proj}(S)$ be a projective scheme over a field k of characteristic $p > 0$. Let M be a $\mathbb{Z}$-graded S-module, and $\mathcal{F}$ the associated sheaf on X. Consider the S-module F_*M. This is not a $\mathbb{Z}$-graded S-module. The goal is to define a $\mathbb{Z}$-graded submodule of F_*M, whose associated sheaf is the quasicoherent sheaf $F_*\mathcal{F}$.

First, we give F_*M a $(1/p)\mathbb{Z}$-grading as follows: For a homogenous element $m \in M$, we define the degree of m considered as an element of F_*M to be $deg(m)/p$. Let M' be the S-submodule $[F_*M]_{\mathbb{Z}}$. In other words, M' is the S-submodule of F_*M generated by homogeneous elements whose degrees in M are multiples of p.

(1) Show that M' is a $\mathbb{Z}$-graded S-module.
(2) Moreover, show that $F_*\mathcal{F} \cong \widetilde{M'}$.

Theorem 4.2.2 (Karen Smith). *Let X be a projective variety over a field k of characteristic $p > 0$. Then the following are equivalent:*

(1) *X is Frobenius split.*
(2) *For all ample line bundles $\mathcal{L}$ on X, the section ring $\bigoplus_{n\geq 0}\Gamma(X, \mathcal{L}(n))$ is Frobenius split.*
(3) *There exists an ample line bundle $\mathcal{L}$ on X such that the section ring $\bigoplus_{n\geq 0}\Gamma(X, \mathcal{L}(n))$ is Frobenius split.*

Proof. I am going to leave this as aa difficult, but still doable exercise. For (iii) $\Rightarrow$ (i), you may want to use the previous exercise. □

Using the previous theorem, one obtains the following corollary:

Corollary 4.2.3. *Let X be a projective scheme over a field of prime characteristic. Then X is Frobenius split if and only if the section ring $\bigoplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(n))$ is Frobenius split.*

Coming back to our original question, we would like to understand how Frobenius splitting of $\mathrm{Proj}(k[X_0, \dots, X_n]/I)$ is related to Frobenius splitting of the graded ring $k[X_0, \dots, X_n]/I$. The previous corollary provides a partial answer. Before we can give this partial answer, we need to discuss Serre's Conditions S_n.

Definition 4.2.4. Let X be a locally Noetherian scheme. Then X satisfies **Serre's condition** S_n if for all $x \in X$,

$$\mathrm{depth}(\mathcal{O}_{X,x}) \geq \min(\dim \mathcal{O}_{X,x}, n).$$

Since for a Noetherian local ring $(A, \mathfrak{m})$, $\mathrm{depth}(A) \leq \dim(A)$ (when equality hols, A is called Cohen-Macaulay), it follows that if X satisfies S_n, then for all $x \in X$ such that $\dim \mathcal{O}_{X,x} \leq n$, $\mathcal{O}_{X,x}$ is Cohen-Macaulay. Thus, X is Cohen-Macaulay if and only if it satisfies S_n for all n.

By far, perhaps the most important is the S_2 condition. For instance, a Noetherian integral domain is normal if and only if it is S_2 and regular in codimension 1. The S_2 condition is also intimately related to the notion of reflexive sheaves. Reflexive sheaves arise naturally as sheaves associated to Weil divisors (in a non-singular setting we don't always get line bundles).

Proposition 4.2.5. *Let $X = \mathrm{Proj}(k[X_0, \ldots, X_n]/I)$ such that $k[X_0, \ldots, X_n]/I$ is $S2$ (for e.g. if $k[X_0, \ldots, X_n]/I$ is a normal domain or Cohen-Macaulay) and has Krull dimension ≥ 2. Then X is Frobenius split if and only if $k[X_0, \ldots, X_n]/I$ is Frobenius split.*

Proof. We have a natural ring homomorphism

$$k[X_0, \ldots, X_n]/I \to \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)).$$

By the assumption that $k[X_0, \ldots, X_n]/I$ is $S2$ of dimension ≥ 2, it follows that the above map is an isomorphism. The easiest way to see this is to use local cohomology, so we will omit the proof. The proposition now follows from the previous corollary. □

Remark 4.2.6. If in the previous proposition $k[X_0, \ldots, X_n]/I$ is normal, then X is projectively normal.

Exercise 4.2.7. Show that the following:

(1) $\mathbb{P}^n_k$ for all $n \geq 1$ is Frobenius split.
(2) Grassmannian varieties are Frobenius split.
(3) If X is a hypersurface in $\mathbb{P}^n_k$ for $n \geq 2$ defined by a homogeneous polynomial f of degree ≥ 1, then X is Frobenius split if and only if $k[X_0, \ldots, X_n]/(f)$ is Frobenius split. *Hint: $k[X_0, \ldots, X_n]/(f)$ is Cohen-Macaulay.*
(4) (For those with background) Normal toric varieties are Frobenius split.
(5) (For those with background) An upper cluster algebra over a field of prime characteristic is Frobenius split [Benito-Muller-Rajchgot-Smith].

4.3. **Fedder's criterion for Frobenius splitting of hypersurfaces.** In the previous subsection, we saw that a hypersurface in $\mathbb{P}^n_k$ ($n \geq 2$) defined by a homogeneous polynomial f is Frobenius split if and only if the ring graded ring $k[X_0, \ldots, X_n]/(f)$ is Frobenius split. So it would be nice to have some sort of algebraic criterion for when rings of the form $k[X_0, \ldots, X_n]/(f)$ are Frobenius split.

One such criterion, which is local in nature, is due to Richard Fedder. Since Frobenius splitting is a local condition for graded F-finite algebras over fields (Proposition 3.1.5), to take advtange of Fedder's local criterion we will assume in this section that all rings are F-finite. In particular, for hypersurfaces in $\mathbb{P}^n_k$ as above, we will assume k is F-finite to ensure F-finiteness of $k[X_0, \ldots, X_n]/(f)$.

We only state Fedder's criterion for hypersurfaces, when in fact there is a version holds for any quotients of regular local rings [F-Purity and Rational Singularity, Theorem 1.12].

Theorem 4.3.1 (**Fedder's criterion**)**.** *Let $(S, \mathfrak{m})$ be an F-finite regular local ring of prime characteristic, and $f \in S$. Then $S/(f)$ is Frobenius split if and only if*

$$f^{p-1} \notin \mathfrak{m}^{[p]}.$$

Notation 4.3.2. For an ideal I of a ring R of prime characteristic, $I^{[p^e]}$ stands for the ideal of R generated by the $p^e - th$ powers of elements of I. In particular, if I is generated by the set $\{i_\alpha : \alpha \in A\}$, then $I^{[p^e]}$ is generated by $\{i_\alpha^{p^e} : \alpha \in A\}$.

Here is how Fedder's criterion can be applied in projective geometry.

Proposition 4.3.3 (**When is a hypersurface Frobenius split?**). *Let X be a hypersurface in $\mathbb{P}^n_k$ $(n \geq 2)$ defined by a homogeneous polynomial $f(X_0, \ldots, X_n) \in k[X_0, \ldots, X_n]$. Suppose k has characteristic p. The following are equivalent:*

(1) *X is Frobenius split.*
(2) *$k[X_0, \ldots, X_n]/(f)$ is Frobenius split.*
(3) *For all $s \notin (X_0, \ldots, X_n)$, $sf^{p-1} \notin (X_0^p, \ldots, X_n^p)$.*

Proof. The equivalence of (1) and (2) was left as Exercise 4.2.7. Let $\mathfrak{m} = (X_0, \ldots, X_n)$. Note $k[X_0, \ldots, X_n]/(f)$ is Frobenius split if and only if

$$(k[X_0, \ldots, X_n]/(f))_{\mathfrak{m}} = k[X_0, \ldots, X_n]_{\mathfrak{m}}/(f)_{\mathfrak{m}}$$

is Frobenius split by Proposition 3.1.5. The equivalence of (2) and (3) now follows by applying Fedder's Criterion to the regular local ring $k[X_0, \ldots, X_n]_{\mathfrak{m}}$. □

The previous proposition gives us supremely checkable way of determining when hypersurfaces are Frobenius split. Here is a list for you to attempt when you are bored.

Exercise 4.3.4. In what follows, $n \geq 2$ and k is F-finite of characteristic $p > 0$. Show that

(1) The union of the coordinate hyperplanes $k[X_0, \ldots, X_n]/(X_0 \ldots X_n)$ is Frobenius split.
(2) $\mathbb{F}_2[X, Y, Z]/(X^3 + Y^3 + Z^3)$ is not Frobenius split.
(3) $k[X, Y, Z]/(XZ - Y^2)$ is Frobenius split (you can do this without Fedder if you wish).
(4) $k[X, Y, Z]/(X^4 + Y^4 + Z^4)$ is never Frobenius split.
(5) (If you are very bored) $\mathbb{F}_3(U, V, W, X)/(U^5 - XY^4 + 2W^2X^3 + V^5)$ is not Frobenius split. (On a serious note see Corollary 4.8.15 for a slick way to see this!)

4.4. **Review of sheaf cohomology results.** Before we discuss surprising vanishing theorems satisfied by Frobenius split varieties, we recap some results we will need in this subsection:

Theorem 4.4.1 (**Cohomology of affine morphisms**). *Let $f : X \to Y$ be an affine morphism of schemes, and $\mathcal{F}$ be a quasicoherent sheaf on X. Then for all $i \geq 0$,*

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

Proof. Follows from Hartshorne Proposition III.8.1, and Exercise III.8.1. Unlike Exercise III.8.2 you do not need X to be Noetherian. Use vanishing of higher cohomology for arbitrary affine schemes (Hartshorne proves this only for Noetherian affine schemes), which is why he has Noetherian hypotheses. □

Theorem 4.4.2 (**Serre Vanishing**). *Let X be a proper scheme over a Noetherian ring A, and $\mathcal{L}$ be an ample line bundle on X. Then for all coherent sheaves $\mathcal{F}$ on X,*

$$H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n) = 0$$

for all $i > 0$ and $n >> 0$.

Proof. Hartshorne Proposition III.5.3. (If you don't know what an ample line bundle is, you can take this as the definition.) □

Theorem 4.4.3 (**Projection Formula**). *Let $f : X \to Y$ be a morphism of schemes (this actually holds for ringed spaces). Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X,*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Proof. Hartshorne Exercise II.5(d). □

Corollary 4.4.4. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then,*

$$f_*f^*\mathcal{E} \cong f_*\mathcal{O}_X \otimes_Y \mathcal{E}.$$

Theorem 4.4.5 (**Serre duality for line bundles on smooth varieties**). *Let X be a non-singular, irreducible projective scheme over a perfect field k (if k is not perfect assume X is smooth instead of non-singular) of dimension n. Let ω_X be the canonical bundle. Then for all line bundles $\mathcal{L}$, and $0 \leq i \leq n$,*

$$H^i(X, \mathcal{L}) \cong H^{n-i}(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{-1})^\vee.$$

The next theorem we are going to state is true only in characteristic 0, with counterexamples given by Raynaud in characteristic p.

Theorem 4.4.6 (**Kodaira Vanishing**). *Let k be a field of characteristic 0 and X be a smooth, irreducible projective scheme over k, with canonical sheaf ω_X. Then for any ample line bundle $\mathcal{L}$ on X,*

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = 0,$$

for all $i > 0$.

Remark 4.4.7. The Kodaira vanishing theorem in fact says more (check Wikipedia). But we will stick to this version.

With this, we are done reviewing all the results we will need to prove vanishing theorems about Frobenius split varieties.

4.5. **Vanishing theorems for Frobenius split varieties.**

Theorem 4.5.1. *Let X be a Frobenius split scheme over $\mathbb{F}_p$. Let $\mathcal{L}$ be a line bundle on X, and suppose that*

$$H^i(X, \mathcal{L}^n) = 0,$$

for some i and all $n >> 0$. Then $H^i(X, \mathcal{L}) = 0$.

Proof. By hypothesis and Lemma 4.1.6 , the map $F^e : \mathcal{O}_X \to F^e_*\mathcal{O}_X$ has a left inverse for all $e > 0$. Hence, tensoring by $\mathcal{L}$, the map

$$\mathcal{L} \to F^e_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}$$

also has a left inverse for all $e > 0$. Now by the projection formula,

$$F^e_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L} \cong F^e_*(\mathcal{O}_X \otimes_{\mathcal{O}_X} (F^e)^*\mathcal{L}) \cong F^e_*((F^e)^*\mathcal{L}),$$

and $(F^e)^*\mathcal{L} \cong \mathcal{L}^{p^e}$ (think in terms of pulling back the transition functions of $\mathcal{L}$ under F^e). Thus, we see that we have morphisms

$$\mathcal{L} \to F^e_*\mathcal{L}^{p^e}$$

with left inverses for all $e > 0$.

Hence $H^i(X, \mathcal{L})$ is a submodule of $H^i(X, F^e_* \mathcal{L}^{p^e})$ for all $e > 0$. Because F^e is an affine morphism (it is the identity map on the underlying spaces), $H^i(X, F^e_* \mathcal{L}^{p^e}) \cong H^i(X, \mathcal{L}^{p^e})$ by Theorem 4.4.1. Thus, $H^i(X, \mathcal{L})$ is a submodule of $H^i(X, \mathcal{L}^{p^e})$ for all $e > 0$. However, $H^i(X, \mathcal{L}^{p^e}) = 0$ for $e >> 0$ by hypothesis. This completes the proof. □

If X is a projective variety over a field k, then Serre vanishing implies that for any ample line bundle $\mathcal{L}$, the higher cohomologies of $\mathcal{L}^n$ vanish for $n >> 0$. But for Frobenius split varieties, we get something stronger courtesy the previous theorem:

Corollary 4.5.2. *Let X be a Froebnius split projective variety over a field of characteristic p. If $\mathcal{L}$ is an ample line bundle on X, then*

$$H^i(X, \mathcal{L}) = 0$$

for all $i > 0$.

Proof. Exercise. □

Although Kodaira vanishing fails for smooth varieties over fields of characteristic p we do get Kodaira vanishing for Frobenius split varieties:

Theorem 4.5.3. *Let X be a non-singular, irreducible, projective scheme over a perfect field k of characteristic p (replace non-singular by smooth if k is not perfect). Suppose X is Frobenius split, and let ω_X be the canonical sheaf. Then for any ample line bundle $\mathcal{L}$ on X,*

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = 0$$

for all $i > 0$.

Proof. Let $d := \dim(X)$. By Serre Vanishing (Theorem 4.4.2), for all $n >> 0$, and $i > 0$,

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = 0.$$

By Serre Duality (Theorem 4.4.5), for all $0 < i \leq d$, and $n >> 0$

$$H^{d-i}(X, \mathcal{L}^{-n}) = 0.$$

By Theorem 4.5.1, for all $0 < i \leq d$,

$$H^{d-i}(X, \mathcal{L}^{-1}) = 0.$$

Then Serre duality again implies that for all $0 < i \leq d$

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = 0.$$

Finally,

$$H^i(X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) = 0$$

for $i > d$ by Grothendieck vanishing for Noetherian spaces. □

Remark 4.5.4. Note Serre Duality, as stated in Theorem 4.4.5, holds more generally for an equidimensional, Cohen-Macaulay, projective scheme over *any* field k (Hartshorne assumes k is algebraically closed). Hence, we get Kodaira vanishing for any projective, equidimensional, Cohen-Macaulay and Frobenius split scheme over *any* field of prime characteristic, with the canonical sheaf ω_X replaced by the dualizing sheaf (denoted ω°_X in Hartshorne III.7) in the statement of Kodaira vanishing.

4.6. **Frobenius splitting along a divisor.** For simplicity, in this subsection let X be an integral scheme over $\mathbb{F}_p$ and D be an effective Cartier divisor (if X is normal you can take D to be an effective Weil divisor). We have a natural inclusion

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D),$$

where $\mathcal{O}_X(D)$ is line bundle (it is a reflexive sheaf if D is a Weil divisor). Applying the functor F_* then gives us an inclusion

$$F_*\mathcal{O}_X \hookrightarrow F_*\mathcal{O}_X(D).$$

Precomposing with the morphism

$$F : \mathcal{O}_X \to F_*\mathcal{O}_X,$$

we then get a morphism

$$\mathcal{O}_X \to F_*\mathcal{O}_X(D).$$

Definition 4.6.1. Let (X, D) be as above. Then X is **Frobenius split along** D if the morphism

$$\mathcal{O}_X \to F_*\mathcal{O}_X(D)$$

has a left-inverse in the category of $\mathcal{O}_X$-modules. We also say that the pair (X, D) is **Frobenius split**.

Remark 4.6.2. We also get morphisms $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ for each $e > 0$ by repeating the above procedure with F^e instead of F.

Exercise 4.6.3. If (X, D) is F-split, then X is F-split. So in particular, all vanishing theorems for F-split schemes holds for F-split pairs.

Exercise 4.6.4 (**Analogue of Exercise 4.1.4, Lemma 4.1.6**)**.** Let (X, D) be as above.

(1) If for some $e > 0$, $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ splits, then (X, D) is F-split. *Hint: Factor this morphism through* $F_*\mathcal{O}_X(D)$.
(2) If (X, D) is F-split, then $\mathcal{O}_X \to F^e_*\mathcal{O}_X(D)$ splits for all $e > 0$.

4.7. **Weak normality and Frobenius splitting.** We probably all know what a normal domain is. But normality can be reinterpreted in the following way geometrically:

Global interpretation of normality: An integral scheme X of finite type over a field k is normal if and only if every finite, birational morphism $Y \to X$ is an isomorphism (here Y is also integral and of finite type over k).

Weak normality, as the name suggests, is a weakening of normality.

Definition 4.7.1 (**Geometric definition**)**.** An integral scheme of finite type over a field k is **weakly normal** if for any integral scheme Y of finite type over k, a finite, birational, bijective morphism $Y \to X$ is an isomorphism.

Clearly, if X is normal then it is weakly normal.

Theorem 4.7.2. *Let X be an integral scheme of finite type over a field k of characteristic p. If X is F-split, then X is weakly normal.*

Proof. [Brion-Kumar, Proposition 1.2.5] □

4.8. **Which smooth curves are Frobenius split?** For us a *curve* is an integral, dimension 1, projective scheme over an arbitrary field k. We will assume k has characteristic $p > 0$. Note we can also assume proper instead of projective in the definition of a curve because projectivity = properness for curves [Stacks Project, Tag 0A26].

Remark 4.8.1. Hartshorne develops most of the theory of curves using the assumption that the ground field is algebraically closed, whereas we assume the ground field is arbitrary. As a reference for the theory of curves over fields not necessarily algebraically closed, I recommend the Stacks Project Chapters on 'Varieties' and 'Algebraic Curves'. Since the theory of curves over arbitrary fields may not be familiar to everyone, I will recall all facts I need.

Recall that any curve C, being projective, has a *dualizing sheaf* ω_C° [Hartshorne, definition on pg 241]. If C is smooth over its ground field (= non-singular when ground field is perfect), then ω_C° coincides with the canonical bundle $\Omega_{C/k}$.

Lemma 4.8.2 (**Serre duality for curves**). *Let C be a curve over any field. Then C is Cohen-Macaulay. In particular, if ω_C° is a dualizing sheaf, then for any line bundle $\mathcal{L}$ on C,*

$$H^i(C, \mathcal{L}) \cong H^{1-i}(C, \omega_C^\circ \otimes_{\mathcal{O}_C} \mathcal{L}^{-1})^\vee,$$

for $i = 0, 1$.

Proof. To see that C is Cohen-Macaulay, note that any dimension 1 Noetherian integral domain is Cohen-Macaulay. Also, C is clearly equidimensional since it has just 1 irreducible component. Therefore, Serre duality holds for line bundles (Remark 4.5.4). This is the second statement of the lemma. □

An important invariant of a line bundle on a curve is its degree. Since our curves are not necessarily non-singular, we define degree in such a way that Riemann-Roch (at least a version of it) becomes a tautology:

Degree of a line bundle: If $\mathcal{L}$ is a line bundle on a curve C, then its **degree** $deg(\mathcal{L})$ is defined as

$$deg(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O}_C).$$ [5]

Remark 4.8.3. Depending on how you have seen the degree of a line bundle on a non-singular curve being defined, our definition may be called the Riemann-Roch theorem.

One advantage of our definition of degree is

Exercise 4.8.4. (**Invariance of degree under base field extension**) Let C be a curve over a field k. Let k' be a field extension of k and $C' = C \times_k k$. Let $\mathcal{L}$ be a line bundle on C and $\mathcal{L}'$ be the pull-back of $\mathcal{L}$ under the canonical map $C' \to C$. Then

$$deg(\mathcal{L}') = deg(\mathcal{L}).$$

[5] For a coherent sheaf $\mathcal{F}$ on a proper scheme X over a field, $\chi(\mathcal{F})$ is the *Euler characteristic* of $\mathcal{F}$, and equals $\Sigma_i(-1)^i h^i(X, \mathcal{F})$.

This exercise may require some background in cohomology. Basically show that the dimensions of cohomology modules is invariant under field extensions (there is a more general statement involving flat base change).

The amazing fact is that amplitude of a line bundle on a curve can be characterized by the degree of the bundle. This may be a familiar fact for non-singular curves over algebraically closed fields, but remember that a curve for us is a projective, integral scheme of dimension 1 over any field.

Theorem 4.8.5. *Let C be a curve with a line bundle $\mathcal{L}$. Then $\mathcal{L}$ is ample if and only if $deg(\mathcal{L}) > 0$.*

Proof. See [Stacks Project, Tag 0B5X]. □

Another invariant of a curve is its arithmetic genus.

Arithmetic genus: The **arithmetic genus** of a curve C over any field, denoted $p_a(C)$, is $1 - \chi(\mathcal{O}_C)$.

Remark 4.8.6. If C is a curve over an arbitrary field k, $H^0(C, \mathcal{O}_C)$ will in general be a finite field extension of k, and not equal to k. This is the reason why the arithmetic genus for arbitrary curves is not just defined as $h^1(X, \mathcal{O}_C)$. In the chapter on 'Algebraic Curves' the Stacks Project often assumes $h^0(C, \mathcal{O}_C) = k$, and they advocate defining the arithmetic genus in general to be the Euler characteristic $\chi(\mathcal{O}_C)$ [See Tag 0BY7].

Remark 4.8.7. There is also the notion of **geometric genus** of a curve, which by definition is $h^0(C, \Omega_{C/k})$. This does not equal the arithmetic genus in general, unless say C is smooth over k, in which case equality follows by Serre Duality since $\Omega_{C/k}$ is then a dualizing sheaf for C.

Exercise 4.8.8. Show that $p_a(C)$ of a curve C in $\mathbb{P}^2_k$ defined by an irreducible homogeneous polynomial of degree $d > 0$ is $(d-1)(d-2)/2$.

A natural question is when the dualizing sheaf ω_C° is a line bundle. The surprising fact is that this can be answered more generally:

Theorem 4.8.9. *Let X be a projective scheme over a field k. The dualizing sheaf ω_X° is a line bundle if and only if X is Gorenstein (i.e. all stalks of the structure sheaf are Gorenstein local rings).*

Proof. The proof of this is beyond the scope of the course. Hartshorne proves this for a local complete intersection [Hart, Theorem 7.11]. □

Remark 4.8.10. If you don't know what a Gorenstein ring is, you can replace Gorenstein by non-singular in the above theorem. In fact, in the affine local case, one can define a Gorenstein local ring to be a Noetherian local ring which is its own dualizing module under local duality. Alternately, a Gorenstein local ring is a Noetherian local ring which has finite injective dimension as a module over itself. For an introduction to local duality, check Mel Hochster's notes on 'Local Cohomology'.

Since ω_C° will be a line bundle for a Gorenstein curve, we then have

Theorem 4.8.11. *Let C be a Gorenstein curve over a field k. Then*

$$deg(\omega_C^\circ) = -2\chi(\mathcal{O}_C) = 2p_a(C) - 2.$$

Proof. By Serre Duality for curves [Theorem 4.8.2],

$$\chi(\omega_C^\circ) = -\chi(\mathcal{O}_C).$$

Then by definition of degree of a line bundle and arithmetic genus, we get the above chain of equalities. □

A simple consequence of the previous theorem and Theorem 4.8.5 is the following characterization of the ammplitude of the dualizing sheaf by the arithmeti genus:

Corollary 4.8.12. *Let C be a Gorenstein curve over a field. The dualizing sheaf ω_C° is ample if and only if the arithmetic genus $p_a(C) \geq 2$.*

Proof. Exercise. □

Whenever the dualizing sheaf of a Gorenstein projective scheme is ample, the scheme cannot be Frobenius split.

Theorem 4.8.13. *Let X be a projective scheme over a field of prime characterisitic of dimension > 0. Suppose X is Gorenstein and equidimensional. If the dualizing sheaf ω_X° is ample, then X is not Frobenius split.*

Proof. Since Gorenstein rings are Cohen-Macaulay, X is also Cohen-Macaulay and equidimensional. Thus, we can apply Serre duality. In particular,

$$h^{dim(X)}(X, \omega_X^\circ) = h^0(X, \mathcal{O}_X) \neq 0.$$

If X is Frobenius split and ω_X° is ample, then

$$h^{dim(X)}(X, \omega_X^\circ) = 0$$

by Corollary 4.5.2. This is a contradiction. □

The theorem implies that Gorenstein curves of genus at least 2 are not Frobenius split.

Corollary 4.8.14. *Let C be a Gorenstein curve over a field of prime characteristic. If the arithmetics genus of C is ≥ 2, then C is not Frobenius split.*

Also, hypersurfaces in $\mathbb{P}^n$ of high enough degree cannot be Frobenius split:

Corollary 4.8.15. *Let X be a hypersurface in $\mathbb{P}_k^n$, for $n \geq 2$ and k a field of prime characteristic, defined by a polynomial f of degree d. If $d \geq n+2$, then X is not Frobenius split.*

Proof. The dualizing sheaf of a hypersurface of degree d is $\mathcal{O}_X(d-n-1)$. So when $d \geq n+2$, the dualizing sheaf is ample. □