Constructing varieties with prescribed Hodge numbers modulo m in positive characteristic

(joint with Matthias Paulsen)

I Introduction

Let X be a smooth projective variety over $k = \bar{k}$.

Def The Hodge numbers of X are

$$h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$$

The <u>Hodge diamond</u> is

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & \cdot^{\cdot^{\cdot}} & & \ddots & \\ h^{n,0} & & \cdots & & h^{0,n} \\ & \ddots & & \cdot^{\cdot^{\cdot}} & \\ & & h^{0,0} & & \end{array}$$

<u>Ex</u> · C :

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array} \qquad \text{(genus } g\text{)}$$

· $\mathbb{P}^2$:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

· $\mathbb{P}^n$:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ \cdot^{\cdot^{\cdot}} & & 1 & & \ddots \\ 0 & & \vdots & & 0 \\ \ddots & & 1 & & \cdot^{\cdot^{\cdot}} \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(Exc III.7.3 in Hartshorne)

Relations:

(1) $h^{0,0}(X) = 1$

(2) (Serre duality)

$$h^{p,q}(X) = h^{n-p,n-q}(X)$$

(Use SD + $\Omega^p_X \times \Omega^{n-p}_X \to \omega_X$)

(3) (Hodge symmetry)

If char $k = 0$, then

$$h^{p,q}(X) = h^{q,p}(X).$$

<u>Thm</u> (Serre, 1958)

If char $k > 0$, there exists a surface X with

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ a & & b & & a \\ & 0 & & 1 & \\ & & 1 & & \end{array}.$$

(Actually can make one over $\mathbb{F}_p$ that lifts to $\mathbb{Z}_p$.)

II Inverse Hodge problem

Q What Hodge diamonds occur?

This is very hard.

Ex

$$\begin{array}{ccccc} & & 1 & & \\ & a & & a & \\ 0 & & 2 & & 0 \\ & a & & a & \\ & & 1 & & \end{array} = \begin{array}{ccc} & 1 & \\ a & & a \\ & 1 & \end{array} \times \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & a & & a & \\ 0 & & 1 & & 0 \\ & a & & a & \\ & & 1 & & \end{array}$$ is not possible (in char 0) if $a>0$.

Idea: the image of $X \to \underline{Alb}_X$ has dimension 1, so X is fibred.

Then $h^{1,1}(X) \geq 2$.

Ex

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 3 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 0 & & 1 & & 0 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 3 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

is not possible by hard Lefschetz (in char 0).

Q What universal (linear, polynomial, ...) relations exist between $h^{p,q}(X)$?

Thm (Kotschick–Schreieder, 2013) If char $k=0$, the only linear relations are SD & HS.

Thm (vDdB, ... 2020)

If char $k > 0$, the only <u>linear</u> relations are SD.

Thm (Paulsen–Schreieder, 2019)

If char $k = 0$, the only <u>polynomial</u> relations are $h^{0,0}(X) = 1$, SD, and HS.

Thm (vDdB–Paulsen, 2020)

If char $k > 0$, then the only polynomial relations are $h^{0,0}(X) = 1$ and SD.

<u>Idea</u> Construct many Hodge diamonds

- (KS13, vDdB20) as linear combinations of Hodge diamonds of varieties
- (PS19, DBP20) as modulo m reductions of Hodge diamonds of varieties.

<u>Ex</u>

- $$\begin{array}{ccccc} & & 1 & & \\ & a & & a & \\ 0 & & 1 & & 0 \\ & a & & a & \\ & & 1 & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & a & & a & \\ 0 & & 2 & & 0 \\ & a & & a & \\ & & 1 & & \end{array} + \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$\left(\sum h^{p,q} x^p y^q\right) z^{\dim X}$$

$$- \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$= [C \times \mathbb{P}^1] + [\mathbb{P}^2] - [\mathbb{P}^1 \times \mathbb{P}^1]$$

$$\begin{array}{ccccc} & & 1 & & \\ & a & & a & \\ 0 & & 1 & & 0 \\ & a & & a & \\ & & 1 & & \end{array} \equiv \begin{array}{ccccc} & & 1 & & \\ & a & & a & \\ 0 & & m+1 & & 0 \\ & a & & a & \\ & & 1 & & \end{array}$$

$$= \mathrm{Bl}_{m-1 \text{ pts}}(C \times \mathbb{P}^1)$$

$$(\operatorname{mod} m).$$

<u>Thm</u> (PS19, DBP20)

Every block

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & \cdot^{\cdot^{\cdot}} & & \ddots & \\ h^{n,0} & & \cdots & & h^{0,n} \\ & \ddots & & \cdot^{\cdot^{\cdot}} & \\ & & h^{0,0} & & \end{array} \in (\mathbb{Z}/m)^{(n+1)^2}$$

satisfying $h^{0,0}$ [redacted] $= 1$, SD, and HS (if char $k = 0$) is the mod n reduction of some $h^{p,q}(X)$.

III Strategy (PS19, DBP20)

<u>Tools</u> (1) Künneth:

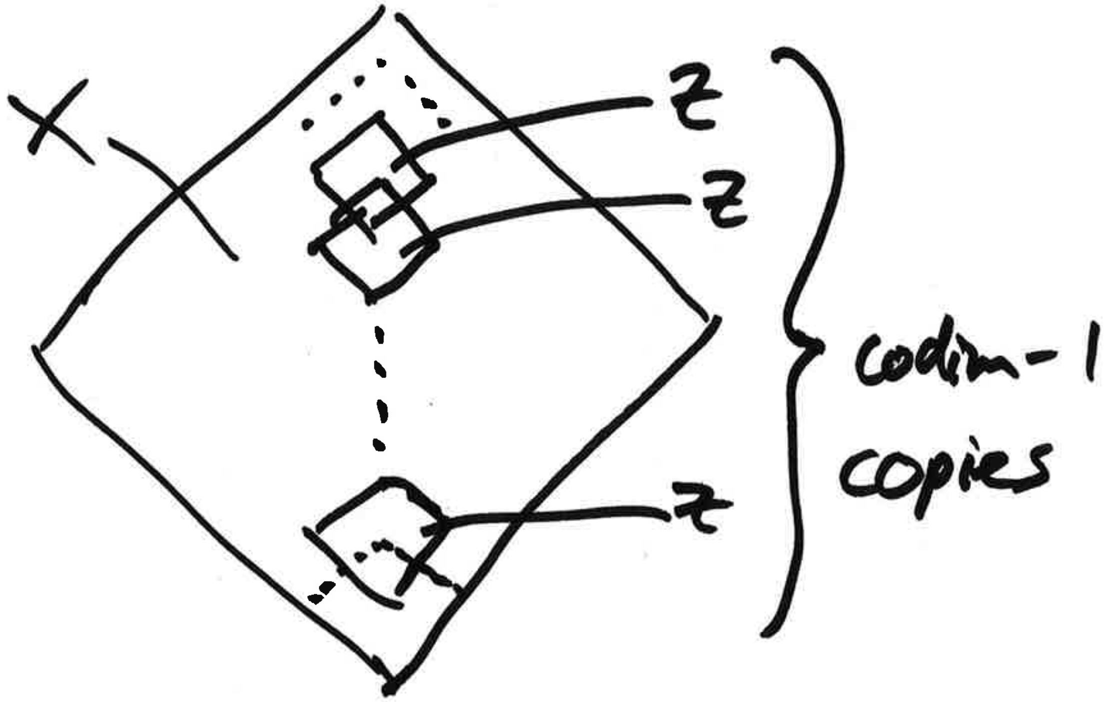
$$\begin{array}{ccc} & 1 & \\ a & & a \\ & 1 & \end{array} \times \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & a & & a & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} + \begin{array}{cccc} & & 0 & \\ & 1 & & 0 \\ a & & a & & 0 \\ & 1 & & 0 \\ & & 0 & \end{array}$$

$$+ \begin{array}{cccc} & & 0 & \\ & 0 & & 1 \\ 0 & & a & & a \\ & 0 & & 1 \\ & & 0 & \end{array} + \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & a & & a & \\ & & 1 & & \end{array}$$

$$= \begin{array}{ccccc} & & [redacted]\ 1 & & \\ & a+1 & & a+1 & \\ a & & 2a+2 & & a\ [redacted] \\ & a+1 & & a+1 & \\ & & 1 & & \end{array}$$

(2) Blowup:

$Bl_Z X$:



Think:

$$Bl_Z X = X - Z + E \approx X - Z + \mathbb{P}^{\text{codim}-1} \times Z.$$

(3) Weak Lefschetz: if $D \subseteq X$ is sufficiently ample, then

Important observation:

Blowing up m isomorphic subvarieties does not change $h^{p,q}$ mod m.

Ex May assume $\mathbb{P}^{n-1} \subseteq X$.
(blow up in m points)

IV Proofs

(1) Outer Hodge numbers by induction. Given

$$\begin{matrix} & & 1 & & \\ \cdots a_1 & & & & b_1 \cdots \\ a_n & & ? & & b_n \end{matrix} \in (\mathbb{Z}/m)^{2n}$$

(in char 0: $a_i = b_i$).

We can already construct

$a_{n-1} = b_{n-1}$

on $(n-1)$-dim. variety X.

Take $X \times \mathbb{P}^2$ and apply WL:

$c = d$

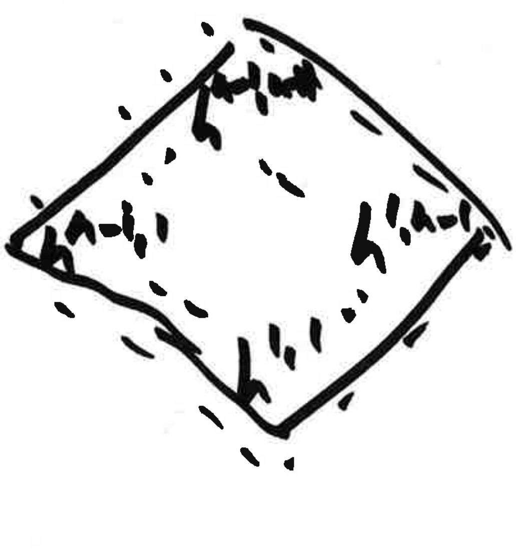
To control c, d choose the divisor appropriately.

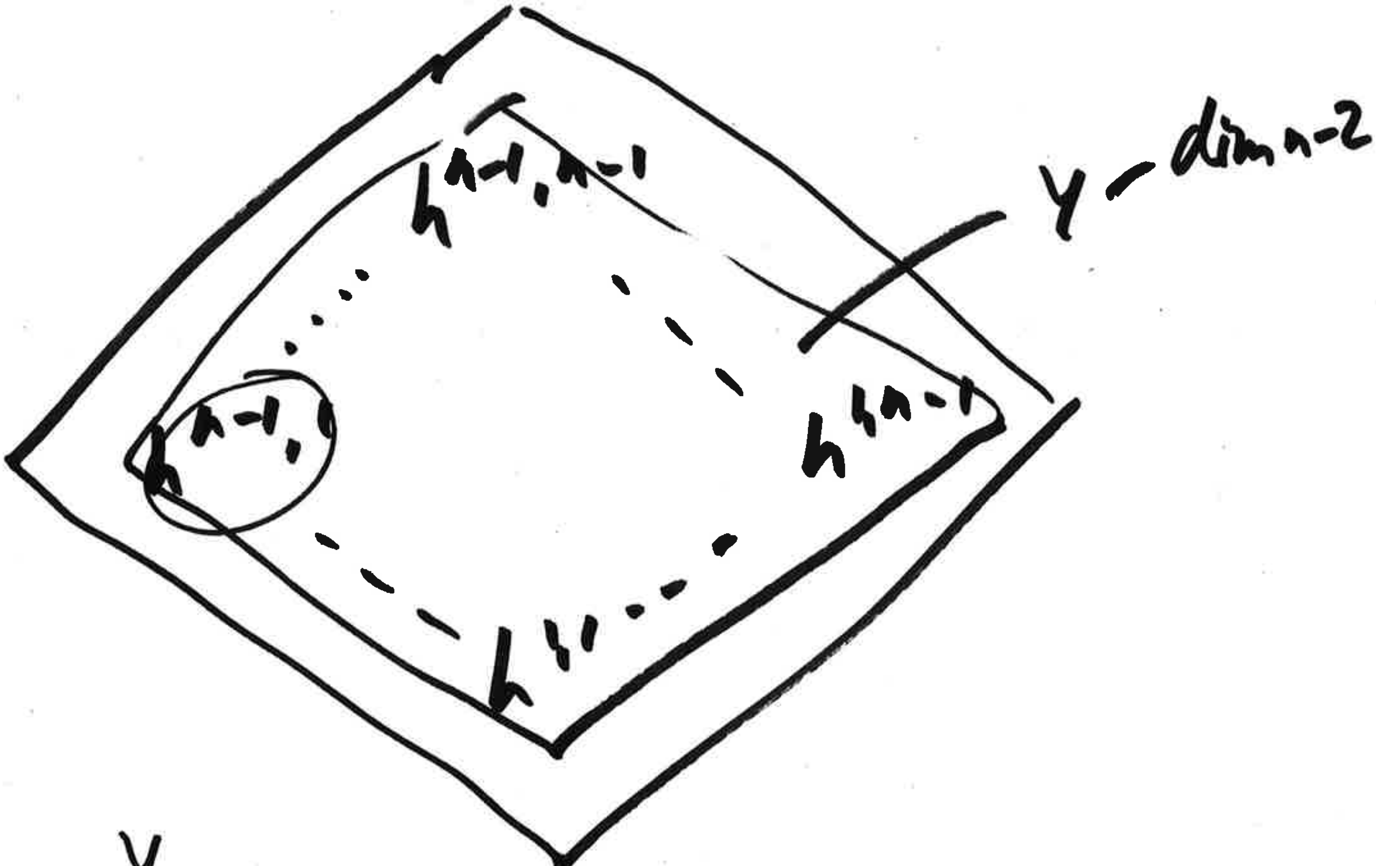
(2) Inner Hodge numbers: blowups.

For simplicity, assume embedded resolution. Given

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & \cdots & & \cdots & \\ h^{n,0} & & \cdots & & h^{0,n} \\ & \cdots & & \cdots & \\ & & h^{0,0} & & \end{array} \in (\mathbb{Z}/m)^{(n+1)^2}$$

we can produce the outer $h'^{p,q}$ on some X.





Now $\underset{\rotatebox{90}{$\in$}\ Y}{\mathbb{P}^{n-1}} \subseteq X$, and

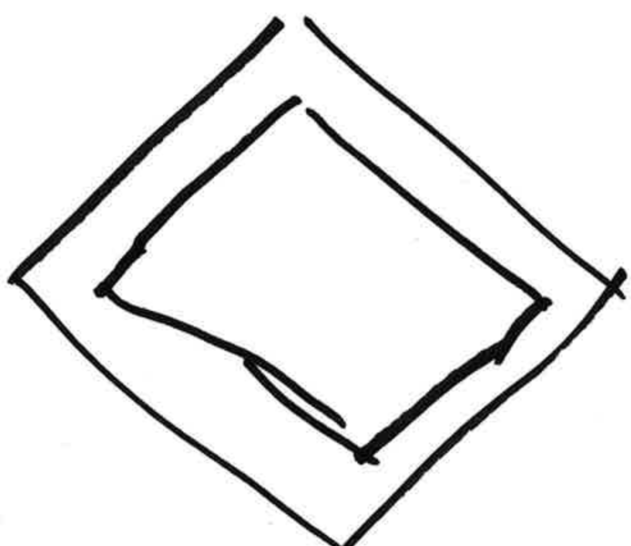
$$Y \overset{\text{bir}}{\sim} \underset{\substack{| \\ \text{singular}}}{Y'} \subseteq \mathbb{P}^{n-1}.$$

Embedded resolution: blow up $\mathbb{P}^{n-1}$ (and X) in smooth centres until $\tilde{Y}' \to Y'$ is smooth.

So we can make a smooth $\tilde{Y}' \supseteq \tilde{X}$.

The outer ~~#~~ $h^{1,1}$ of $\tilde{Y}'$ agree with $h^{1,1}$ of Y.

Blow up $\tilde{X}$ in $\tilde{Y}'$ to get the next edge correct.



$\square$