# A THEOREM OF KUNZ USING PERFECT RINGS

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## 1. Introduction

Throughout, we fix a prime number $p > 0$, and unless otherwise specified, all rings will have characteristic $p$. We will often repeat these hypotheses in the statements of definitions and results. Kunz's characterization of regular rings in characteristic $p$ is the starting point of using the Frobenius map to study singularities. His celebrated theorem states the following:

**Theorem 1.0.1.** *Let $R$ be a Noetherian ring of characteristic $p$. Then $R$ is regular if and only if the Frobenius endomorphism $F_R : R \to R$ is a flat map.*

The *goal* of this note is to discuss a recent proof of the above theorem, given by Bhatt and Scholze, using *perfect rings*. Before introducing perfect rings, we make some standard reductions.

Both regularity and flatness can be checked locally. Moreover, since the Frobenius map $F_R$ induces the identity map on $\operatorname{Spec}(R)$, it suffices to assume in the proof of Theorem 1.0.1 that $R$ is a Noetherian local ring. Moreover, we may also assume that $R$ is complete, via the following result:

**Lemma 1.0.2.** *Let $(R, \mathfrak{m}) \xrightarrow{\varphi} (S, \eta)$ be a local homomorphism of Noetherian local rings. Then $\varphi$ is flat if and only if the induced map on completion $\widehat{R}^{\mathfrak{m}} \xrightarrow{\widehat{\varphi}} \widehat{S}^{\eta}$ is flat.*

The superscript in the above completion denotes the ideals with respect to which one is completing. The above Lemma is somewhat tricky to prove. In the situation of interest to us when $\varphi$ is the Frobenius map $F_R$, one can easily check that $\widehat{\varphi}$ is the Frobenius map $F_{\widehat{R}}$. This allows us to reduce to proving Theorem 1.0.1 for a Noetherian, complete local ring.

Once we have reduced to the complete, local case, since our rings are equicharacteristic (all our rings contain the field $\mathbb{F}_p$), we can use Cohen's structure theorem. In particular, if $R$ is a complete, regular local ring with residue field $\kappa$, then

$$R \cong \kappa[[x_1, \ldots, x_d]],$$

for $d = \dim(R)$. Using this observation, it is not hard to prove one direct of Kunz's theorem, which we leave as an exercise.

**Exercise 1.0.3.** Suppose $R = \kappa[[x_1, \ldots, x_d]]$, for a field $\kappa$ of prime characteristic. Then show that the Frobenius map on $R$ is flat. Hint: Lemma 1.0.2 allows you to reduce to proving flatness of Frobenius for $\kappa[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$.

**Strategy of proof:** In the rest of the note, we will prove the reverse implication. Our strategy will be to use flatness of $F_R$ to find a faithfully flat embedding

$$R \hookrightarrow S$$

of $R$ into a much larger, non-Noetherian ring $S$ which has finite Tor-dimension. Then $R$ will have finite Tor-dimension as well (since the embedding is faithfully flat), and we will be done by the Auslander-Buchsbaum-Serre criterion of regularity of Noetherian local rings.

## 2. Perfect rings- basic constructions

**Definition 2.0.1.** A ring $R$ of characteristic $p$ is **perfect** if the Frobenius map $F_R$ is an isomorphism.

**Exercise 2.0.2.** Let $R$ be a perfect ring. Then show the following:

(1) $R$ is reduced.
(2) If $S \subset R$ is a multiplicative set, then $S^{-1}R$ is perfect.
(3) If $R$ is a local ring with a finitely generated maximal ideal $\mathfrak{m}$, then $R$ is a field.
(4) If $R$ is Noetherian, then it is isomorphic to a finite direct product of perfect fields.

The exercise shows that Noetherian perfect rings are not particularly interesting. Nevertheless, perfect rings arise naturally even when one is interested in only Noetherian rings. For example, if $R$ is a Noetherian domain of prime characteristic with fraction field $K$, then the integral closure of $R$ in an algebraic closure $\overline{K}$ of $K$ is a perfect ring of prime characteristic. This is called the *absolute integral closure of* $R$, usually denoted $R^+$. An important result of Hochster and Huneke says that if $R$ is an excellent local domain of characteristic $p$, then $R^+$ is a *big Cohen-Macaulay algebra.*

There are two natural ways to produce perfect rings from any ring of prime characteristic, as highlighted in the next exercise:

**Exercise 2.0.3.** Let $R$ be a ring of characteristic $p$. Then we have the following:

(1) The ring $R_{perf}$ defined to be the colimit of

$$R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \dots$$

is a perfect ring. It is universal with respect to maps from R to perfect rings (interpret what this means).
(2) The ring $R^{perf}$ defined to be the limit of

$$\dots \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} R$$

is a perfect ring universal with respect to maps from a perfect ring to R.

In these notes we are mostly interested in the $R_{perf}$ construction, although $R^{perf}$ is crucial in the theory of perfectoid spaces.

**Examples 2.0.4.**

(1) $(\mathbb{F}_p)_{perf} = \mathbb{F}_p = (\mathbb{F}_p)^{perf}$.
(2) If $x$ is an indeterminate, then $\mathbb{F}_p[x]_{perf} = \mathbb{F}_p[x^{1/p^e} : e \in \mathbb{N}] =: \mathbb{F}_p[x^{1/p^\infty}]$.
(3) If $R$ is a reduced ring and $R^{p^e}$ denotes to the image of the $e$-th iterate of the Frobenius map (the subring of $p^e$-th powers of elements of $R$), then

$$R^{perf} = \bigcap_{e>0} R^{p^e}.$$

Thus $\mathbb{F}_p[x]^{perf} = \mathbb{F}_p$.
(4) If $R$ is a domain, then $R_{perf}$ embeds in the absolute integral closure $R^+$.

**Lemma 2.0.5.** *Let $R$ be a perfect ring of characteristic $p$. An ideal $I$ of $R$ is radical if and only if there exists a subset $S \subset R$ such that*

$$I = \langle f^{1/p^\infty} : f \in S \rangle.$$

*Proof.* Suppose $I$ is a radical ideal of $R$. If $S$ is a generating set of $I$, then clearly

$$I = \langle f^{1/p^\infty} : f \in S \rangle.$$

Conversely, suppose $I = \langle f^{1/p^\infty} : f \in S \rangle$. For $e > 0$, if

$$I_e \coloneqq \langle f^{1/p^e} : f \in S \rangle,$$

then it is easy to verify that $I$ is the union of the chain of ideals $I_e$. Suppose $r \in \sqrt{I}$. Then there exists $f > 0$ such that $r^{p^f} \in I$. In particular, there also exists $e > 0$ such that $r^{p^f} \in I_e$. Taking $p^f$-th roots, it follows that

$$r \in I_{e+f} \subset I.$$

□

**Remark 2.0.6.** If $R$ is a perfect ring, then a quotient $R/I$ of $R$ is perfect precisely when $I$ is a radical ideal of $R$, and so has the form as given in Lemma 2.0.5.

## 3. Homological properties of perfect rings

Since we are mostly interested in analyzing Tor functors over perfect rings, we will use homological numbering conventions for complexes (this may be more familiar to the audience). Throughout this section, $R$ will denote a perfect ring of prime characteristic $p > 0$.

### 3.1. Derived tensor products.

Given $R$-modules (here $R$ can be any ring) $M$ and $N$,

$$M \otimes^{\mathbb{L}}_R N$$

denotes the complex $F_\bullet \otimes_R N$, where $F_\bullet$ is a flat (e.g. projective) resolution of $M$. Thus,

$$H_i(M \otimes^{\mathbb{L}}_R N) = \operatorname{Tor}_i^R(M, N).$$

The complex $M \otimes^{\mathbb{L}}_R N$ is the *derived tensor product* of $M$ and $N$, and lives in the derived category $D(R)$. More generally, given complexes $I_\bullet$ and $J_\bullet$ of $R$-modules, one can form the derived tensor product

$$I_\bullet \otimes^{\mathbb{L}}_R J_\bullet$$

by taking a *$K$-flat resolution* $F_\bullet \to I_\bullet$ of $I_\bullet$ (it's not important to know what a $K$-flat resolution of a complex is), and then defining $I_\bullet \otimes^{\mathbb{L}}_R J_\bullet$ to be the complex $F_\bullet \otimes_R J_\bullet$.

Derived tensor products make it easier to keep track of the behavior of Tor-modules under change of rings, because for the most part one can reason with them just as one would with 'ordinary' tensor products. One does need to know anything about $D(R)$ in what follows, except the fact that if two complexes of R-modules are isomorphic in $D(R)$, then their homologies are isomorphic.

**Proposition 3.1.1.** *Here are some properties of derived tensor products used in the sequel. All isomorphisms are in $D(R)$.*

1. $I_\bullet \otimes^{\mathbb{L}}_R J_\bullet \simeq J_\bullet \otimes^{\mathbb{L}}_R I_\bullet$ *and* $(I_\bullet \otimes^{\mathbb{L}}_R J_\bullet) \otimes^{\mathbb{L}}_R L_\bullet \simeq I_\bullet \otimes^{\mathbb{L}}_R (J_\bullet \otimes^{\mathbb{L}}_R L_\bullet)$.
2. *If $N$ is a flat $R$-module,* $M \otimes^{\mathbb{L}}_R N \simeq M \otimes_R N$. [1]
3. *If $S$ is an $R$-algebra and $M, N$ are $R$-modules, then* $(M \otimes^{\mathbb{L}}_R N) \otimes^{\mathbb{L}}_R S \simeq (M \otimes^{\mathbb{L}}_R S) \otimes^{\mathbb{L}}_S (N \otimes^{\mathbb{L}}_R S)$.
4. *If $S$ is a flat $R$-algebra, then* $(M \otimes^{\mathbb{L}}_R N) \otimes^{\mathbb{L}}_R S \simeq (M \otimes_R S) \otimes^{\mathbb{L}}_S (N \otimes_R S)$.

[1] We think of modules as complexes concentrated in degree 0.

3.2. **Tor-vanishing for perfect rings.** The main result of this section, from which the proof of Kunz's theorem will follow without too much effort is the following:

**Theorem 3.2.1** (Bhatt-Scholze)**.** *Let $R$ be a perfect ring of characteristic $p$ and $S, T$ be perfect $R$-algebras. Then*

$$S \otimes_R^{\mathbb{L}} T \simeq S \otimes_R T,$$

*that is* $\mathrm{Tor}_i^R(S,T) = 0$*, for all* $i > 0$*.*

We will have to prove some preparatory results before getting to the proof of this theorem. The next proposition is going to do all of the heavy-lifting.

**Proposition 3.2.2.** *Let $R$ be a perfect ring of characteristic $p$. Let $f \in R$ (not necessarily a non-zerodiviso r) and $I$ be the ideal $(f^{1/p^\infty})$.*

1. $I = \mathrm{colim}(R \xrightarrow{f^{\frac{p-1}{p}}} R \xrightarrow{f^{\frac{p-1}{p^2}}} R \xrightarrow{f^{\frac{p-1}{p^3}}} R \longrightarrow \dots)$*. Consequently, $I$ is a flat $R$-module.*
2. *If $S$ is a perfect $R$-algebra, then the canonical map*

$$I \otimes_R S \to S$$

   *mapping $i \otimes s \mapsto is$ is injective.*
3. *If $S$ is a perfect $R$-algebra, then*

$$S \otimes_R^{\mathbb{L}} R/I \simeq S \otimes_R R/I = S/IS.$$

4. *Let $f_1, \dots, f_n \in R$. Then for any perfect $R$-algebra $S$,*

$$S \otimes_R^{\mathbb{L}} R/(f_1^{1/p^\infty}, \dots, f_n^{1/p^\infty}) \simeq S \otimes_R R/(f_1^{1/p^\infty}, \dots, f_n^{1/p^\infty}).$$

*Proof.* (1) Consider the commutative diagram

$$\begin{array}{ccccccc} R & \xrightarrow{f^{\frac{p-1}{p}}} & R & \xrightarrow{f^{\frac{p-1}{p^2}}} & R & \xrightarrow{f^{\frac{p-1}{p^3}}} & R \dots \\ \downarrow f & & \downarrow f^{1/p} & & \downarrow f^{1/p^2} & & \downarrow f^{1/p^3} \\ fR & \hookrightarrow & f^{1/p}R & \hookrightarrow & f^{1/p^2}R & \hookrightarrow & f^{1/p^3}R \dots \end{array}$$

Let $\mathcal{D}$ be the colimit of the top row; it is a flat $R$-module since a filtered colimit of flat $R$-modules is flat. The colimit of the bottom row is clearly $I$. Since the vertical maps are all surjective, we have a surjective $R$-linear map

$$\eta : \mathcal{D} \twoheadrightarrow I.$$

Thus to prove (1), it suffices to show $\eta$ is an isomorphism. Then consider the commutative square

$$\begin{array}{ccc} R & \xrightarrow{f^{\frac{p-1}{p^{n+1}}}} & R \\ \downarrow f^{1/p^n} & & \downarrow f^{1/p^{n+1}} \\ f^{1/p^n}R & \hookrightarrow & f^{1/p^{n+1}}R \end{array}$$

Suppose $r$ is an element of the top left copy of $R$ in the above diagram such that

$$f^{1/p^n} r = 0$$

in $f^{1/p^n}R$. Since $R$ is perfect and reduced, taking $p$-th roots we then get

$$f^{p-1/p^{n+1}} r = f^{p-2/p^{n+1}} (f^{1/p^n} r)^{1/p} r^{p-1/p} = 0.$$

This clearly shows that $\eta$ must be injective, hence an isomorphism.

(2) By (1) and the fact that tensor products commute with filtered colimits, we get

$$I \otimes_R S \cong \operatorname{colim}(S \xrightarrow{f^{\frac{p-1}{p}}} S \xrightarrow{f^{\frac{p-1}{p^2}}} S \xrightarrow{f^{\frac{p-1}{p^3}}} \to \dots) \cong IS.$$

The second isomorphism follows because $S$ is also a perfect ring, and so one can repeat the argument in (1) for $S$ instead of $R$. It is easy to verify that the composition of the above chain of isomorphisms equals the natural map

$$I \otimes_R S \to IS$$

which sends $i \otimes s \mapsto is$.

(3) By (1), $0 \to I \hookrightarrow R \to 0$ is a flat resolution of $R/I$. Thus

$$\operatorname{Tor}_i^R(S, R/I) = 0$$

for all $i \geq 2$. Moreover, (2) implies that

$$\operatorname{Tor}_1^R(S, R/I) = \ker(I \otimes_R S \to S) = 0.$$

This precisely means that $S \otimes_R^{\mathbb{L}} R/I \simeq S \otimes_R R/I$.

Finally, (4) follows from (3) and an easy induction argument. □

The aim is to use the previous proposition to prove Theorem 3.2.1 by reducing to the case where the perfect $R$-algebra $T$ is a quotient of $R$. For this, we need the following simple lemma.

**Lemma 3.2.3.** *Let $A \to B \to C$ be ring homomorphisms such that $A \to B$ is flat. Then for any $A$-module $M$,*

$$M \otimes_A^{\mathbb{L}} C \simeq (M \otimes_A B) \otimes_B^{\mathbb{L}} C.$$

*Thus $\operatorname{Tor}_i^B(M \otimes_A B, C) \cong \operatorname{Tor}_i^A(M, C)$, for all $i \geq 0$.*

*Proof.* Since $B$ is a flat $A$-algebra, we have

$$(M \otimes_A B) \otimes_B^{\mathbb{L}} C \simeq (M \otimes_A^{\mathbb{L}} B) \otimes_B^{\mathbb{L}} C \simeq M \otimes_A^{\mathbb{L}} (B \otimes_B^{\mathbb{L}} C) \simeq M \otimes_A^{\mathbb{L}} C.$$

The assertion about Tor now follows by taking homology. □

**Lemma 3.2.4.** *Let $\varphi : R \to S$ be a homomorphism of perfect rings. Then $\varphi$ factors as*

$$R \to T \xrightarrow{\widetilde{\varphi}} S$$

*where $T$ is perfect, $R \to T$ is flat and $\widetilde{\varphi}$ is surjective.*

*Proof.* By choosing a presentation of $S$ as an $R$-algebra, one can always choose a surjection from a polynomial ring

$$R[X_i : i \in I] \twoheadrightarrow S$$

onto $S$ that extends $\varphi$. Since $S$ is perfect, by the universal property of $R[X_i]_{perf}$, we then get a surjective homomorphism

$$\widetilde{\varphi} : R[X_i]_{perf} \twoheadrightarrow S$$

extending $\varphi$. Let $T \coloneqq R[X_i]_{perf}$. Since $R[X_i]$ is a flat $R$-algebra and $T$ is a flat $R[X_i]$-algebra, it follows that $T$ is a flat $R$-algebra. □

We are finally ready to prove Theorem 3.2.1.

***Proof of Theorem 3.2.1.*** We are given perfect $R$-algebras $S, T$, where $R$ is a perfect ring. In order to prove Theorem 3.2.1, it suffices to assume that $T$ is a quotient of $R$. Indeed, when $T$ is not a quotient of $R$, one can always factor the structure map

$$R \to T = R \to T' \twoheadrightarrow T,$$

where $T'$ is a flat, perfect $R$-algebra by Lemma 3.2.4. Then by Lemma 3.2.3, we have

$$S \otimes^{\mathbb{L}}_R T \simeq (S \otimes_R T') \otimes^{\mathbb{L}}_{T'} T \simeq (S \otimes_R T') \otimes_{T'} T = S \otimes_R T,$$

where the second quasi-isomorphism follows from our knowledge of the theorem when one of the involved algebras is a quotient.

After assuming that $T = R/I$, since $I$ must be radical, it follows that $I$ is a filtered limit of the ideals of the form

$$(f_1^{1/p^\infty}, \dots, f_n^{1/p^\infty} : f_1, \dots, f_n \in I).$$

Therefore $T$ is a filtered direct limit of quotient rings of the form $R/(f_1^{1/p^\infty}, \dots, f_n^{1/p^\infty})$, and since derived tensor products (as well as tensor products) commute with filtered direct limits, it suffices to prove the theorem when $T = R/(f_1^{1/p^\infty}, \dots, f_n^{1/p^\infty})$. But then we are done because of Proposition 3.2.2(4). □

3.3. **Tor-dimension of quotients of perfect rings.** Recall that a ring $R$ (not necessarily Noetherian) has Tor**-dimension** $\leq d$ if every $R$-module $M$ has a flat resolution of length at most $d$. Equivalently, for all $R$-modules $M, N$ and $i > d$, $\mathrm{Tor}_i^R(M, N) = 0$, that is,

$$M \otimes^{\mathbb{L}}_R N$$

is acyclic in degrees $> d$. If such a $d$ exists, then we say that $R$ has **finite Tor-dimension**.

A famous result of Auslander-Buchsbaum-Serre says that Noetherian local rings of finite Tor-dimension are precisely regular local rings. However, quotients of rings of finite Tor-dimension rarely have finite Tor-dimension. For example, quotients of regular local rings need not be regular local rings.

As a consequence of Tor-vanishing for perfect rings, something completely unexpected happens for quotients of perfect rings that are also perfect.

**Proposition 3.3.1.** *Let $R \twoheadrightarrow S$ be a surjective ring homomorphism of perfect rings of characteristic $p$. Then we have the following:*

(1) $S \otimes^{\mathbb{L}}_R S \simeq S$.
(2) *For any $S$-module $M$, $M \otimes^{\mathbb{L}}_R S \simeq M$.*
(3) *For $S$-modules $M, N$, $M \otimes^{\mathbb{L}}_S N \simeq M \otimes^{\mathbb{L}}_R N$.*
(4) *If $R$ has finite* Tor*-dimension, then $S$ has finite* Tor*-dimension.*

*Proof.* (1) By Theorem 3.2.1

$$S \otimes^{\mathbb{L}}_R S \simeq S \otimes_R S \cong S,$$

where because $R \to S$ is surjective, the multiplication map $S \otimes_R S \to S$ sending $s_1 \otimes s_2 \mapsto s_1 s_2$ is an isomorphism.

(2) $M \simeq M \otimes^{\mathbb{L}}_S S \simeq M \otimes^{\mathbb{L}}_S (S \otimes^{\mathbb{L}}_R S) \simeq (M \otimes^{\mathbb{L}}_S S) \otimes^{\mathbb{L}}_R S \simeq M \otimes^{\mathbb{L}}_R S$.

(3) $M \otimes^{\mathbb{L}}_R N \simeq (M \otimes^{\mathbb{L}}_S S) \otimes^{\mathbb{L}}_R N \simeq (M \otimes^{\mathbb{L}}_S (S \otimes^{\mathbb{L}}_R S)) \otimes^{\mathbb{L}}_R N \simeq (M \otimes^{\mathbb{L}}_S S) \otimes^{\mathbb{L}}_R (S \otimes^{\mathbb{L}}_R N)$. But now since $M, N$ are $S$-modules, by (2) we get

$$(M \otimes^{\mathbb{L}}_S S) \otimes^{\mathbb{L}}_R (S \otimes^{\mathbb{L}}_R N) \simeq M \otimes^{\mathbb{L}}_R N.$$

Finally (4) is an easy consequence of (3) because if $R$ has Tor-dimension $\leq d$, then $M \otimes^{\mathbb{L}}_S N \simeq M \otimes^{\mathbb{L}}_R N$ will be acyclic in degrees $> d$. □

## 4. Proof of Kunz's Theorem

We want to show that if $(R, \mathfrak{m}, \kappa)$ is complete, local Noetherian ring such that $F_R : R \to R$ is flat, then $R$ is regular. By Cohen's structure theorem, we may assume that

$$R = \kappa[[x_1, \ldots, x_n]]/I.$$

Since $F_R$ is faithfully flat, the $R$-algebra $R_{perf}$ is faithfully flat. Thus, if we can show that $R_{perf}$ has finite Tor-dimension, then $R$ will have finite Tor-dimension, and consequently $R$ will be regular. Here we are using the following general fact:

**Lemma 4.0.1.** *Let $R \to S$ be a faithfully flat ring homomorphism. If $S$ has Tor-dimension $\leq d$, then $R$ has Tor-dimension $\leq d$.*

*Proof.* Let $M, N$ be $R$-modules. We have to show that $M \otimes^{\mathbb{L}}_R N$ is acyclic in degrees $> d$. On the other hand, we know that $(M \otimes_R S) \otimes^{\mathbb{L}}_S (N \otimes_R S)$ is acyclic in degrees $> d$. But

$$(M \otimes_R S) \otimes^{\mathbb{L}}_S (N \otimes_R S) \simeq (M \otimes^{\mathbb{L}}_R N) \otimes_R S.$$

Since $S$ is faithfully flat, the desired acyclicity assertion for $M \otimes^{\mathbb{L}}_R N$ now follows. □

Coming back to the situation at hand, $R_{perf}$ is a quotient of the perfect ring $\kappa[[x_1, \ldots, x_n]]_{perf}$. Thus, if $\kappa[[x_1, \ldots, x_n]]_{perf}$ has finite Tor-dimension, then $R_{perf}$ will also have finite Tor-dimension by Proposition 3.3.1. But $\kappa[[x_1, \ldots, x_n]]_{perf}$ has Tor-dimension $\leq n$ by the following lemma, and so we are done with the proof of Kunz's theorem.

**Lemma 4.0.2.** *Let $(A_i, \varphi_{ij})_{i,j \in I}$ be a filtered direct system of rings such that each transition map $\varphi_{ij} : A_i \to A_j$ is flat. Suppose there exists $d \in \mathbb{N}$ such that each $A_i$ has* Tor-*dimension $\leq n$. Then*

$$A \coloneqq \operatorname{colim}_{i \in I} A_i$$

*has* Tor-*dimension $\leq n$.*

*Proof.* Since the transition maps $\varphi_{ij}$ are flat, it is not difficult to verify that for all $i \in I$, $A$ is a flat $A_i$-algebra. Let $M, N$ be A-modules. Let $F_\bullet \to M$ be a flat resolution of $M$ consisting of $A$-modules. Then $F_\bullet$ is also a flat resolution of $M$ as $A_i$-modules. Thus,

$$M \otimes^{\mathbb{L}}_A N \simeq F_\bullet \otimes_A N = \operatorname{colim}_{i \in I} F_\bullet \otimes_{A_i} N \simeq \operatorname{colim}_{i \in I} M \otimes^{\mathbb{L}}_{A_i} N.$$

Each $M \otimes^{\mathbb{L}}_{A_i} N$ is acyclic in degrees $> n$ because $A_i$ has Tor-dimension $\leq n$. Since homology commutes with filtered colimits, this implies $M \otimes^{\mathbb{L}}_A N$ is acyclic in degrees $> n$. □

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