A topological proof of the Shapiro–Shapiro Conjecture

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Parametric curves and Wronskians

- Parametric curve $\phi : \mathbb{P}^1 \to \mathbb{P}^k$:

$$t \mapsto \phi(t) = [\ f_0(t) : \cdots : f_k(t)\ ],\ \text{where } f_i(t) \in \mathbb{C}[t]_{\leq n}.$$

- The **Wronskian** of $f_0, \ldots, f_k$ is given by

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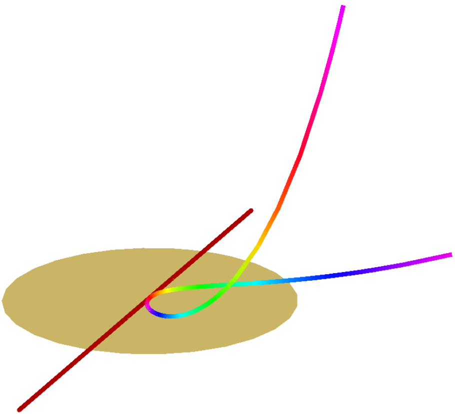
- Detects **flexes**: t such that $\phi, \phi', \phi'', \ldots, \phi^{(k)}$ is linearly dependent (e.g. inflection point, cusp, ...)
  - **Simple flex**: Rank deficiency at $\phi^{(k)}$, fixed at $\phi^{(k+1)}$.

A higher-dimensional simple flex (in $\mathbb{P}^3$)



- C meets its tangent line to order 2 (generic behavior)
- C meets its tangent plane to order $\mathbf{3}+\mathbf{1} = \mathbf{4}$ (flex!)

Real and complex flexes

Example (Critical points of rational functions)

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Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be given by

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\end{aligned}$$

The Wronskian computes the **critical points** where $\phi'(t) = 0$:

$$\mathrm{Wr}(\phi) = \det \begin{bmatrix} f_0 & f_1 \\ f_0' & f_1' \end{bmatrix} = f_0 f_1' - f_0' f_1 = 1 - t^4.$$

4 critical points at $t = \pm 1, \pm i$.

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Basic combinatorial question: how many?

Theorem (Classical)

There are only finitely-many parametric curves ϕ with flexes at prescribed $t_i \in \mathbb{P}^1$ (up to PGL_{k+1}).

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Deep connection to Schubert calculus:

The number of such curves (counted with multiplicity) is the number of **standard Young tableaux**:

$$\mathrm{SYT}(\square\!\square\!\square / \square\!\square\!\square) = \left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}\ , \ldots \right\}$$

Over $\mathbb{R}$, things are remarkably nice!

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Theorem (Mukhin–Tarasov–Varchenko '05, '09)

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Goal for today:

Theorem (L–Purbhoo '19)

Let $\mathrm{Wr}(\phi)$ have n_1 distinct real roots, n_2 complex conjugate pairs.

Over $\mathbb{R}$, the number of such ϕ, counted with signs, is the symmetric group character $\chi^{\ydiagram{3,3}}(2^{n_2}, 1^{n_1})$.

Recovers M–T–V in the case $n_2 = 0$.

(Wronskians and) Schubert calculus

The **Grassmannian** is the space of planes:

$$\mathrm{Gr}(k, \mathbb{C}^n) = \{\text{vector subspaces } S \subset \mathbb{C}^n : \dim(S) = k\}.$$

For us: subspaces $\langle f_0, \ldots, f_k \rangle$ of the space of polynomials $\mathbb{C}[t]_{\leq n}$.

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Simplest (codimension 1) **Schubert variety**:

$$X^{\square}(F_{n-k}) = \{S \in \mathrm{Gr}(k, n) : S \cap F_{n-k} \neq 0\}.$$

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$$\mathscr{F} : \mathbb{C}^n = F_n \supset F_{n-1} \supset \cdots \supset F_1.$$

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For us: "divisibility flags" $\mathscr{F}(z)$ for $z \in \mathbb{C}$:

$$\mathscr{F}(z) : \{f \text{ divisible by } (t-z)\} \supset \{f \text{ divisible by } (t-z)^2\} \supset \cdots .$$

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Gives the **Wronski map**:

$$\mathrm{Wr} : \mathrm{Gr}(k+1, \mathbb{C}[t]_{\leq n}) \to \mathbb{P}(\mathbb{C}[t]_{\leq (k+1)(n-k)}),$$
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- **Fiber** of the Wronski map $\Leftrightarrow$ set of ϕ with specified flexes.
- $\mathrm{Wr}(\phi) = (t - t_1) \cdots (t - t_N)$
  $\iff t_1, \ldots, t_N$ are flexes of the curve $\phi(t)$.

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This is an intersection of Schubert varieties:

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Schubert calculus: Such an intersection is counted (with multiplicity) by **standard Young tableaux**:

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Shapiro–Shapiro / M–T–V, geometric statement

Theorem (Mukhin–Tarasov–Varchenko '05, '09)

If $\mathrm{Wr}(\phi)$ *has all real roots, then the fiber is reduced and every point in it is real.*

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- and more (Purbhoo, Halacheva–Rybnikov–Kamnitzer–Weeks, L–Gillespie, White, . . .)

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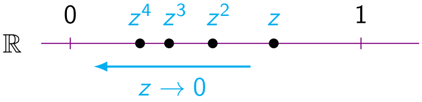
Take $\lim_{z \to 0}$.



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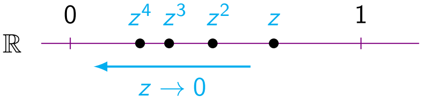

What will happen to the fiber at $z = 0$?

Critical points of rational functions, redux

Degree-3 maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, $\phi(t) = \dfrac{a_3 t^3 + a_2 t^2 + a_1 t + a_0}{b_3 t^3 + b_2 t^2 + b_1 t + b_0}$:

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  - Wronskian has 4 roots z, z^2, z^3, z^4, critical points of ϕ.
  - Fiber $=$ two points, counted by $\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \, , \, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \right\}$.

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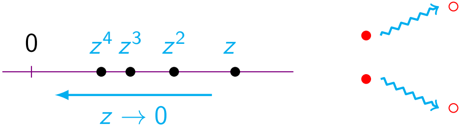

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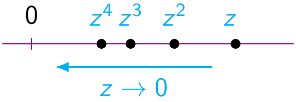

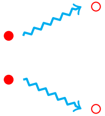

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Tableau labels from Plücker coordinates

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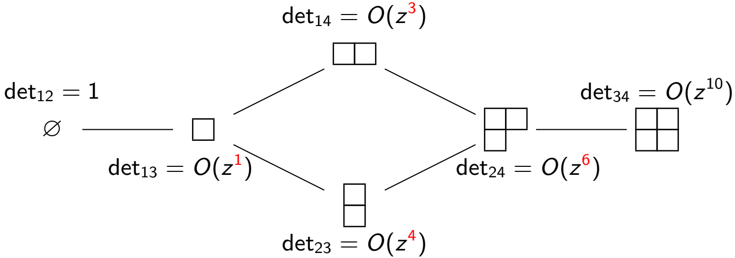

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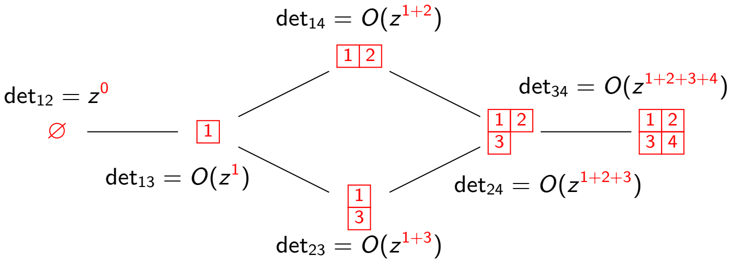

Combinatorics and geometry

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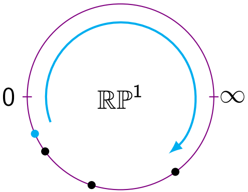
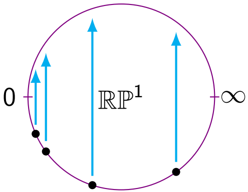


tableau promotion



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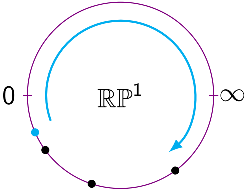

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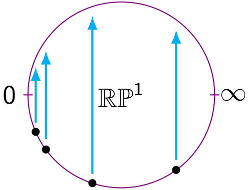

tableau evacuation

And more:

- Topology and genus when $\dim(Z) = 1$ (L, Gillespie–L)
- Orthogonal Grassmannians (Purbhoo, Gillespie–L–Purbhoo)
- Vector bundles on $\overline{M_{0,n}}$ (Kamnitzer, Rybnikov)

A challenge and a new approach

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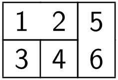
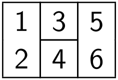
Now: conjugate roots in $\mathbb{C}$ and a topological approach.

(−) 

, (+) 

, …

Oriented Young tableaux.

Generalization: complex conjugate roots for $\mathrm{Wr}(\phi)$

Definition (Cutting up $\mathbb{R}[t]_{\leq N}$)

For a partition $\mu = (2^{n_2}, 1^{n_1})$, let $P(\mu)$ be

$$P(\mu) = \left\{ \text{polynomials with } \begin{array}{l} n_1 \text{ distinct real roots,} \\ n_2 \text{ complex conjugate pairs} \end{array} \right\}$$
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Base case: $\mu = (1^N)$, all real roots.

- We study the restricted Wronski map

$$\mathrm{Wr}_\mu : \mathrm{Wr}^{-1}(P(\mu)) \to P(\mu).$$

Note: we're always in the Schubert cell $X^{\emptyset}(\infty)^{\circ}$ (no roots at ∞).

Topological and algebraic degrees

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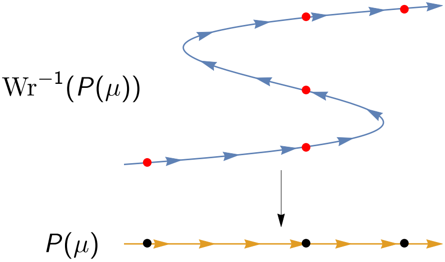

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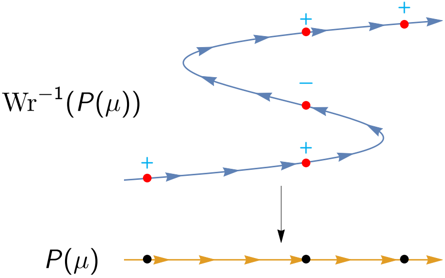

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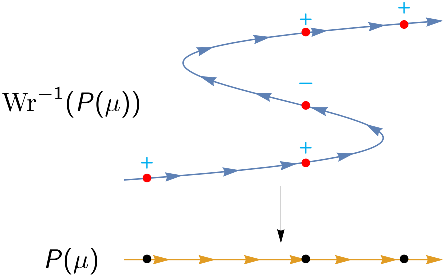

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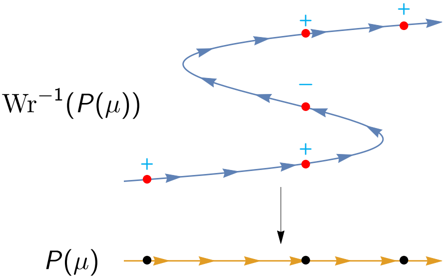

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- We use a new **"character" orientation** on the Schubert cell.

The topological degree of Wr_μ

Character table of S_4. $(\chi^\lambda(\mu))$

λ, μ	(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)
(4)	1	1	1	1	1
$(3,1)$	1	0	-1	-1	3
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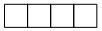	1	1	1	1	1
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Under the **character orientation**, *the restricted Wronski map* Wr_μ *has topological degree* χ(μ).

Signed Young tableaux

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Signed Young tableaux

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Murnaghan–Nakayama rule for $\chi^\lambda(\mu)$, $\mu = (2^{n_2}, 1^{n_1})$:

$$\chi^\lambda(\mu) = \sum_T (-1)^{\#\square(T)} : \begin{array}{l} \mu\text{-domino tableaux } (+)\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array},\ (-)\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline\end{array}, \cdots \\ \text{shape}(T) = \lambda. \end{array}$$

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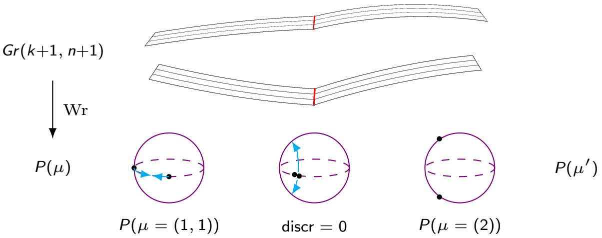
- Special case: $\mu = (1^N)$, no dominos $\rightsquigarrow \chi^{\square\square\square}(1^N) = \#\mathrm{SYT}$.
- **Corollary**: Shapiro–Shapiro Conjecture.

Character orientation of $Gr(k+1, n+1)$

- Boundary between different $P(\mu)$'s when roots collide:

$$\Delta = \{\mathrm{discr}(\mathrm{Wr}) = 0\} \subset \mathbb{R}[t]_{\leq N}.$$



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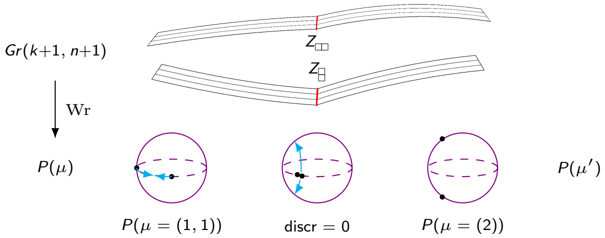

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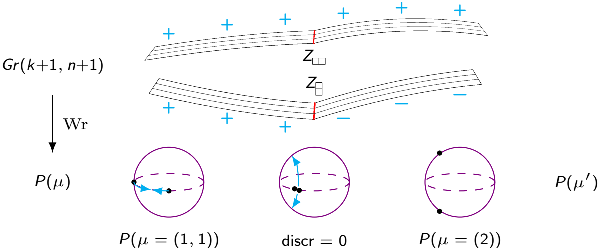

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  - Type $\square\square$: rank deficiency at $\phi^{(k)}$ and again at $\phi^{(k+1)}$.
- **Character orientation**: multiply by the equation of $Z_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$.

Labeling fibers by signed Young tableaux

Proof sketch (signs agree with M–N rule):

- Label limit fibers by tableaux.
- Track $+/-$ signs along a **network of paths**: $\begin{smallmatrix}\square\\\square\end{smallmatrix} \leftrightarrow (-1)$.

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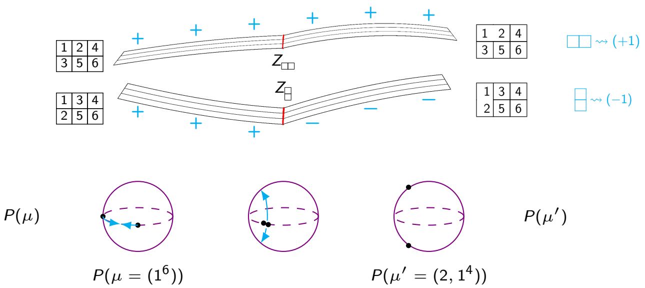
Case 1: $\begin{smallmatrix}1\\2\end{smallmatrix} \longleftrightarrow \begin{smallmatrix}1\\2\end{smallmatrix}$ / $\boxed{1\,|\,2} \longleftrightarrow \boxed{1\;2}$



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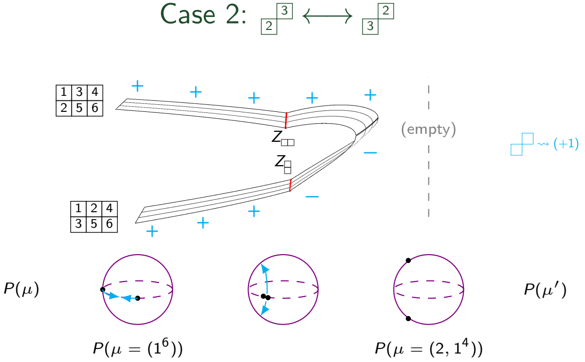

Open questions

- (Representation theory).
  Do all S_N character values $\chi^\lambda(\mu)$ give topological degrees of real Schubert problems? ($\mu \neq (2^a 1^b)$)

- (Complex geometry).
  Explicit geometry of Wr_μ over $P(\mu)$ for $\mu \neq (1^N)$?

- (Stable curves).
  How does the geometry look over the moduli space $\overline{\mathcal{M}_{0,N}}$?
  - $\overline{\mathcal{M}_{0,N}}(\mathbb{R})$ is non-orientable!

Many interesting relationships to find between geometry and combinatorics.

Thank you!